INTERSECTIONALITY

VANCOUVER, APRIL 2014

Exhibit Catalogue

ALANA BOLTWOOD

Metadata

Artist

My name is Alana Boltwood, and as an artist, I call myself a "meta-geometer". I create pictures and things that are *about geometry,* about mathematics, about electronics. My art is also about the shape of our society: politics, cultures and economics.

In the *Intersectionality* series, you will see glimpses of my original career as a statistician. Now (when not in the art studio) I am a management consultant, specializing in public-sector information management (see www.metimea.ca).

I live in Toronto, with roots in Ottawa and Vancouver. My training is largely from the Toronto School of Art. I have exhibited paintings, collages and assemblages around the city, and performed interventions at Toronto's *Nuit Blanche* all-night art extravaganzas. Before getting into the visual arts, my creative energies burst out in the kitchen, the pottery studio, and the choir loft.

You can see more of my art at www.meta-geometer.com.

Exhibit

The art exhibition entitled *Intersectionality* was shown at the opening night event for the international conference *Intersectionality Research, Policy and Practice: Influences, Interrogations and Innovations*, held at Simon Fraser University, April 24-26, 2014, in Vancouver.

Permissions

INTERSECTIONALITY

INTRODUCTION

In grade school math, I learned about Venn diagrams and the intersection of sets. When I learned to design databases, there were cross-references called intersection entities. And when I delved into queer activism, everybody talked about intersectionalities with other oppressed groups.

These disciplines use very different methods, but they share the term "intersection" with mathematical set theory. Whether or not I understand those specialized languages, I can see artistic compositions in their symbols, and their texts prompt free-association. Un-intended new meanings can be found in the mash-up of academic concepts.

While marking and assembling these materials, I've explored my own confusion about being privileged yet marginalized, in a state known as hybrid intersectionality. Identity politics separates people into oversimplified binary categories that don't explain my position in society. Sometimes I feel guilty for having a more successful career than most women, regardless of my gender non-conformity. Sometimes I'm angry about the struggle to fit in socially; I consider explanations ranging from my non-monogamy to my eczematous skin. This mid-life crisis project took me right back to childhood, when the analysis was simply: I'm different.

The intersectionality theory leads us to recognize that individuals are unique, but they can form coalitions in the struggle against oppression. Maybe I became a statistician, survey researcher and definition-writer to do the complex analyses required to prepare us for that struggle. Or maybe I just enjoy the beauty of language and of mathematics.

But data analysis cannot express my feelings – delighted or distraught. And so, I have made art about math and marginalization, statistics and skin, categorization and conformity, power and truth.

Intersection of Contents

This collage was the work that began the *Intersectionality* series.

Diagrams of intersecting sets are suggested by the stripes of text, taken from the Tables of Contents of two works: Kunen's mathematical set theory textbook[1] alternating with Ehrenreich's law-journal article about intersectionality[2].

The multi-coloured background is "skin tissue" alluding to the rainbow of diversity. Each dot in the background symbolizes a person, who may be a member of zero, one or more sets. The pointy ovals block our view of the people in society's intersections.

This perforated metal disc used to outline these circles, and imprint dots on the pointy ovals, has somehow been used to make every other piece in the series.

---

[1] Kunen, Kenneth, *Set Theory: An Introduction to Independence Proofs (*North Holland - Elsevier Science Publishers, 1980).

[2] Ehrenreich, Nancy, "Subordination and symbiosis: mechanisms of mutual support between subordinating systems," *UMCK Law Review* 71(2003):2.

Intersection of Contents, 18"x14", acrylic & paper on canvas, 2010

HYBRID INTERSECTIONALITY (SELF-PORTRAIT)

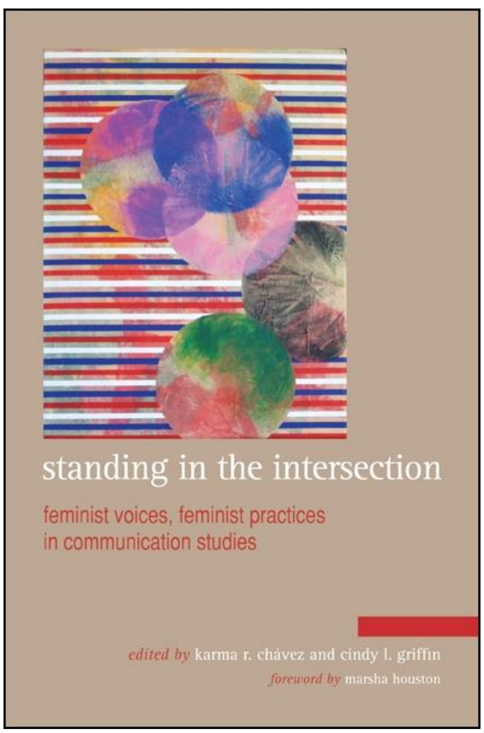
This painting illustrates the concept of hybrid intersectionality[3], in which a person or group is both privileged and oppressed.

The red, white and blue stripes represent my privilege of speaking English and being a Canadian of British and French extraction.

The intersecting coloured circles represent my more oppressed group identities. Women today are assigned a feminine pink. The purple, pink, blue and black colours are found in various queer symbols. The green alludes to environmentalism and paganism.

The background texture from *Intersection of Contents* reappears, but with brighter reds. This mottled "skin" is now more personal; something as apolitical as a red skin rash can be cause for oppression.

This painting became the cover image on an anthology:



Chávez, K.R. and C.L. Griffin (eds), *Standing in the Intersection: Feminist Voices, Feminist Practices in Communication Studies* (SUNY Press, 2012).

[3] Ehrenreich, 2003.

Hybrid Intersectionality (Self Portrait), 18"x14", acrylic & paper on canvas, 2011.

Six Set Venn Diagram

This painting illustrates a Venn diagram[4] designed by A.W.F. Edwards to show all the possible intersections of six sets. The design was for a stained-glass window in memory of John Venn.

(Today the term "Venn diagram" is commonly applied to any diagram of intersecting sets, but if only some of the possible logical relations are shown, "Euler diagram" is the correct term.)

We can see herein the difficulty of illustrating the intersection of many oppressed groups. The more oppressions apply to a person, the harder it is to see them when looking at the whole society. This is not just metaphorical: random sampling often can't find a statistically-valid sample of people in the rarer intersections, so how do you publish a credible study?

[4] Wikipedia contributors, "Venn Diagram," *Wikipedia, The Free Encyclopedia*, http://en.wikipedia.org/wiki/Venn_diagram (accessed March 30, 2014).

Six Set Venn Diagram, 18"x14", acrylic on canvas, 2013

HOW DISABILITY AFFECTS FRIENDSHIP BETWEEN WOMEN

The six-set Venn diagram reappears, to frame collaged text that summarizes an article[5] about the intersection between womanhood and dis/ability.

A selection of the text:

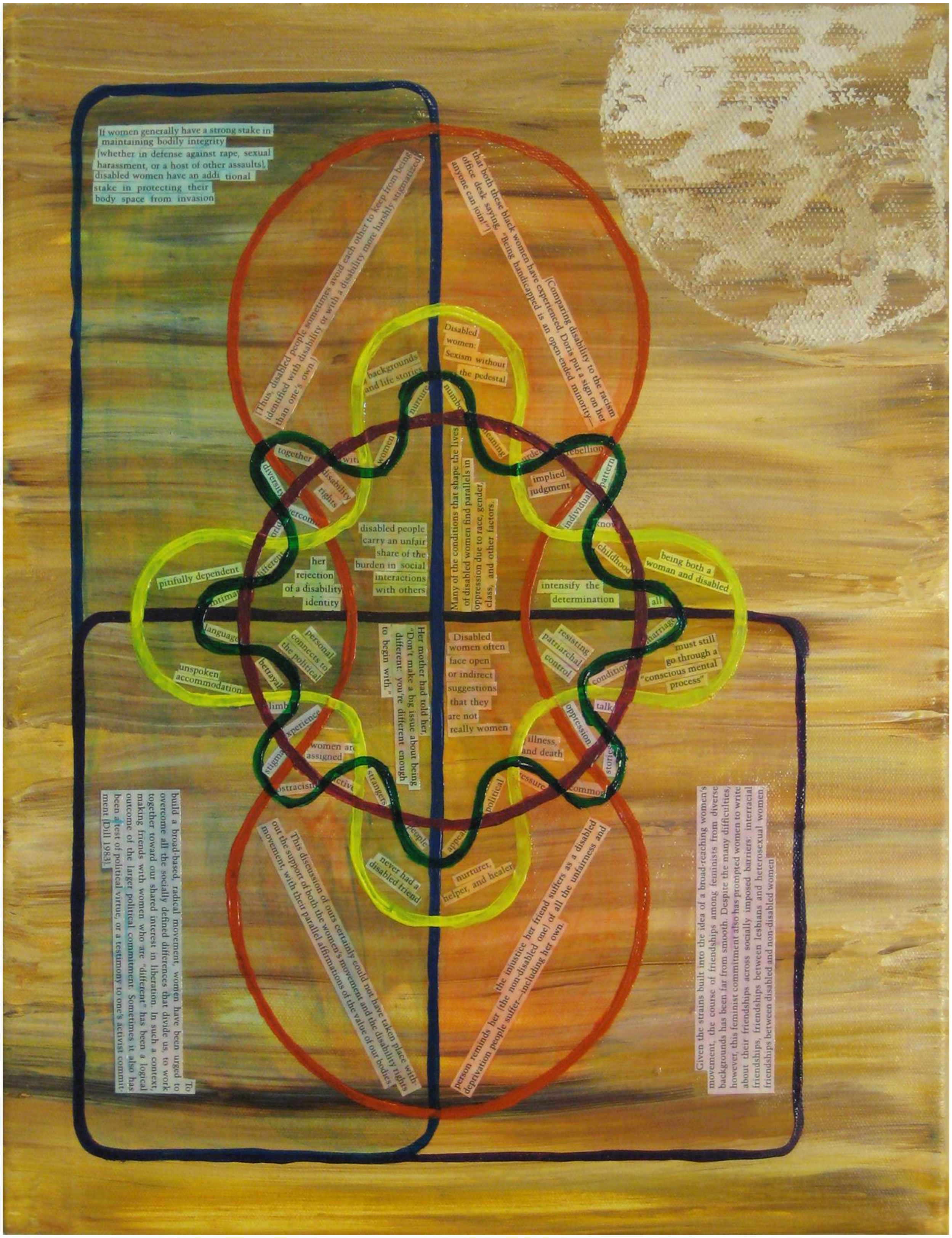
…the women's movement and
the disability rights movement,
with their parallel affirmations of
the value of our bodies.

…making friends with women who are "different"
has been a logical outcome of
the larger political commitment.
Sometimes it also has been a test of political virtue,
or a testimony to one's activist commitment.

…the injustice her friend suffers as a disabled person
reminds her (the non-disabled one) of
all the unfairness and deprivation people suffer –
including her own.

Disabled women: Sexism without the pedestal

[5] Fisher, Berenice and Roberta Galler, "Friendship and fairness: How disability affects friendship between women", in*Women with Disabilities*, ed. Michelle Fine and Adrienne Asch (Temple University Press, 1988), 172-194.



How Disability Affects Friendship Between Women, 18"x14", acrylic & paper on canvas, 2013

Change Can Come Through Insider Resistance

The title and text of this painting are adapted from this quotation of a sociology textbook[6] about Collins' *Black feminist thought*[7]:

> In this domain, change can come through insider resistance.
> Collins uses the analogy of an egg.
> From a distance,
> the surface of the egg looks smooth and seamless.
> But upon closer inspection,
> the egg is revealed to be riddled with cracks.
> For those interested in social justice,
> working in a bureaucracy is like working the cracks,
> finding spaces and fissures to work and expand.
> Again, change is slow and incremental.

My experience working in the public sector confirms that bureaucratic change is slow and incremental.

My experience as an activist is that changing the policies of government, or large businesses, can also be done by thinking like an insider: Finding fissures where a little policy change will make a big difference, and pushing (in gentle but firm bureaucratic language) to open up that crack.

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[6] Allan, Kenneth D., "Chapter 16 Web Byte-Patricia Hill Collins", online supplement to *Contemporary Social and Sociological Theory: Visualizing Social Worlds* (Sage Publications, 2013), accessed March 30, 2014, http://www.sagepub.com/upm-data/13299_Chapter_16_Web_Byte_Patricia_Hill_Collins.pdf

[7] Collins, Patricia Hill, *Black feminist thought: Knowledge, consciousness, and the politics of empowerment* (Routledge, 2000), 2nd edition.

Change Can Come Through Insider Resistance, 18"x14", acrylic & gesso on canvas, 2013

FEMINIST COMPLEXITY THEORY

This collage of Japanese papers is marked with quotes from an article by Gottfried[8] about the intersectional oppression of women who migrate to Japan to do reproductive labour (nannies, maids, etc.)

The limitation of most intersectionality scholarship to a United States or Western focus leads Gottfried to discuss sociological methods for understanding the full complexity of gender, class, race and nation.

To model the intersection of those and more dimensions of privilege/oppression, it is also plausible to use mathematical approaches, ranging from multivariate statistics, to computational complexity theory (represented by the cluster of lines in the centre).

A selection of the quotations on the collage (some of which are abbreviated, and some of which are from work of other authors (Acker, Lenz, McCall, and Mohanty, as cited by Gottfried):

Feminist theories of intersectionality
need to pay more attention to intra-categorical differences
as well as inter-categorical relationships

There were no ungendered class relations
and no gender without class dimensions

Transmigrants, transcultural or trans-desiring/queer persons –
the 'trans' perspective become more important,
but it cannot be integrated into the
concept of essentialized homogenous groups

Complexity of social relationships between social inequalities
cannot be studied as if contained within national borders

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[8] Gottfried, Heidi, "Reflections on Intersectionality: Gender, Class, Race and Nation," ジェンダー研究 (*Journal of Gender Studies,* Ochanomizu University), 11(2008):23-40, accessed March 30, 2014, http://www.igs.ocha.ac.jp/igs/IGS_publication/journal/11/jenda_2_heidi.pdf

Feminist Complexity Theory, 18”x14”, paper, ink & acrylic on canvas, 2013

MARGINALIA

This collage is made from marginalia by an anonymous reader, on a copy of *Disposable Domestics*[9] found in a used bookstore in San Francisco's Mission District.

The reader noted that one of the most salient points in the text is:

> …efforts were made to direct black women
> to these same homes as domestic workers.
> The overall result was to create altogether separate
> standards and conditions under which
> women of color and white women had to mother…

To reflect on this work about labour, I turned to micro-economics[10]. The total product of labour is the basis for graphing the variable cost curve, as illustrated in the collage. The slope of this curve (mathematically, its derivative) is the marginal cost, so I annotated it with sloped strips of marginalia.

---

[9] Chang, Grace, *Disposable domestics: immigrant women workers in the global economy* (South End Press, 2000).

[10] Rosenman, Robert. "From Production to Cost", notes for Intermediate Microeconomics 301 at Washington State University, accessed March 29, 2014. http://faculty.ses.wsu.edu/rosenman/dist301/costs2.htm

Marginalia,18”x14”, acrylic & paper on canvas, 2014

BINARIES

Ehrenreich[11] writes that "Each identity gains meaning through its contrast with other identities."

Inside electronic[12] devices, all information is reduced to the binary system of zeros and ones. People have to be categorized and labelled for computers to understand them.

In our glamourously[13] automated world, how can we break free of the assumption that everyone is either male or female? How do we trouble[14] the digital gender binary?

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[11] Ehrenreich, 2003.

[12] Green elements are from editorial and advertising in *EDN: The Design Magazine of the Electronics Industry*, October 17, 2002, September 25, 2003 and July 24, 2003.

[13] Pink and red elements are from editorial and advertising in *Glamour*, October 2013.

[14] Butler, Judith. *Gender Trouble: Feminism and the Subversion of Identity* (Routledge, 1990).

Binaries,18”x14”, acrylic & paper on canvas, 2014

Axes of Power

This painting uses political and statistical symbols to explore the artist's personal questions: Why do I enjoy privilege and success in this capitalist democracy, somehow escaping the typical oppressions of gender-variant women and bisexual people? Why do I encounter social marginalization that can't be explained by being in some identity group?

The dark-coloured background symbolizes various political economies: royalty, democracy, militarism, capitalism, industrialism, agricultural feudalism, hippie communes, communist totalitarianism, violent chaos, unorganized subsistence. The horizontal black axis indicates the amount of group control while the vertical black axis indicates the amount of individual freedom.

The light-coloured foreground is a graph (suggesting a statistical factor-analysis[15] plot) of people within those societies: what factors determine whether an individual is influential or insignificant?

The near-horizontal white line shows the trend that many individuals are privileged or oppressed according to the power of the demographic and identity groups that the individual belongs to.

The white line at a 45-degree angle shows another cluster of people: those who succeed or fail according to their own capabilities and efforts, regardless of their demographics. Some people are affected by a combination of the demographic and individual factors, so they float away from either cluster.

There is a quotation by Michel Feher[16], from an art-theory book, collaged into the painting:

> Defining them as audiences strictly in terms of gender, ethnicity or region
> is precisely the conceptual problem.
> In fact the artificial definition of communities that are divided among themselves
> is the conceptual disease of the whole left post-modernist moment.

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[15] Johnson, Richard A, *Applied Multivariate Statistical Analysis* (Prentice-Hall, 1988).

[16] Foster, Hal, ed., *Discussions in Contemporary Culture, No. 1* (Dia Art Foundation, 1987), 166.

Axes of Power, 56"x72", acrylic & paper on canvas, 2013

Ehrenreich Monoprints

Each of these uniquely made prints illustrates a concept from Ehrenreich's article[17], as specified by the print's title. Irregular loops are indented in the paper to represent the irregularly-bounded sets of people with shared interests.

Each image includes set-theory terminology[18] in mirror-image, selected through an obscure thought process inspired by Ehrenreich's concepts, rather than by mathematical meaning.

- *Exclusion* is inked in a magenta reminiscent of gay culture. The spotted imprint is the blue used for wheelchair symbols. The set theory phrase is "forcing with finite partial functions".
- *Battle of Oppressions* uses the green and grey of military uniforms, and blood red from the battlefield. The set theory phrase is "relativization".
- *Reinforced Subordination* is inked in a light brown skin tone of racialized people, with an imprint of the gay magenta. The set theory phrase is "reflection theorems".
- *Zero-Sum Problem* is inked in the wheelchair-symbol blue, with a balance inked in dollar-bill green. The set theory phrase is "embeddings, isomorphisms".
- *Divide & Conquer* is inked in feminine pink, and imprinted with a dark brown skin tone. The set theory phrase is "classes and recursion".
- *Infinite Regress* is inked in all the above colours to symbolize all the identity groups. The set theory phrase is "almost disjoint & quasi-disjoint sets".

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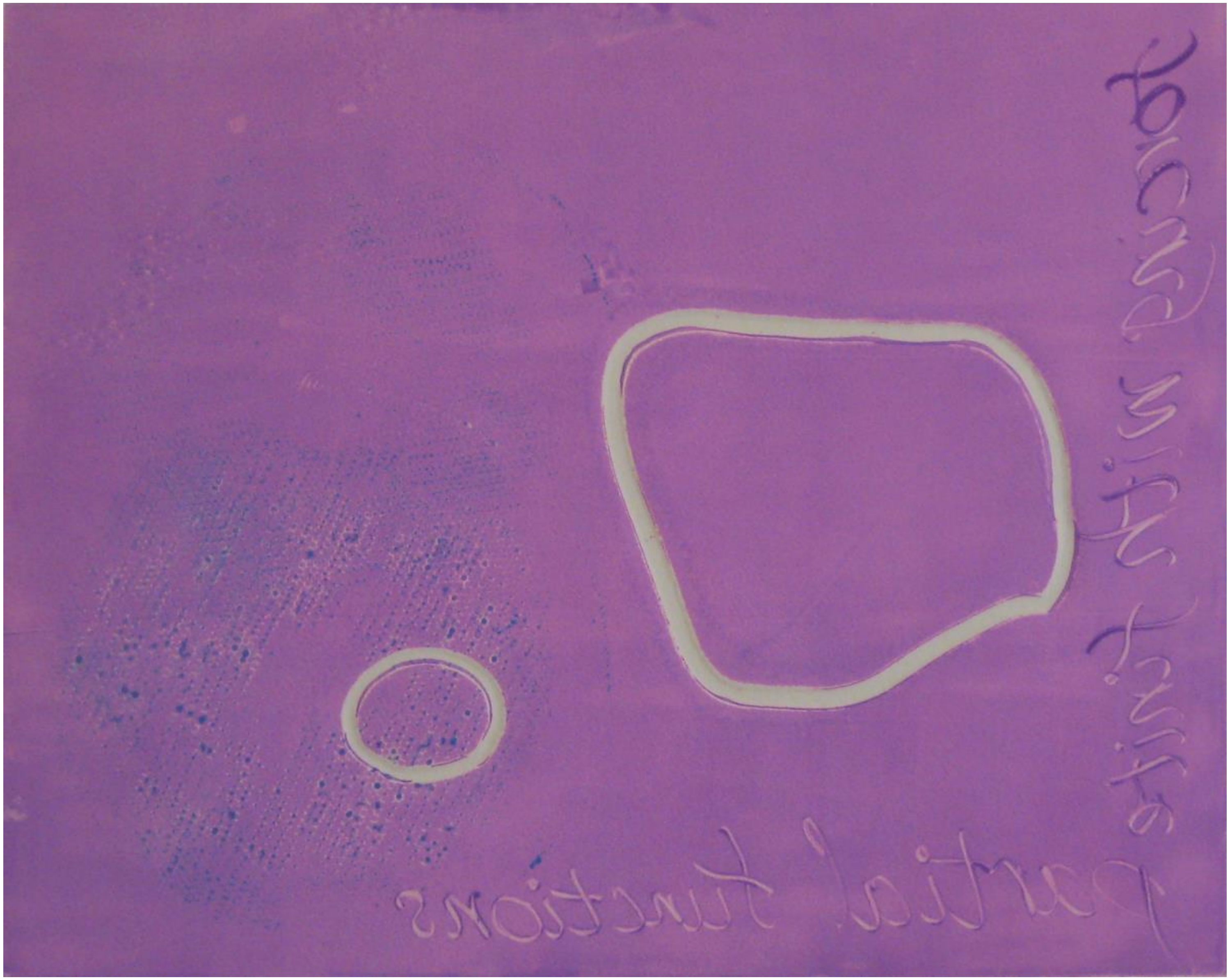
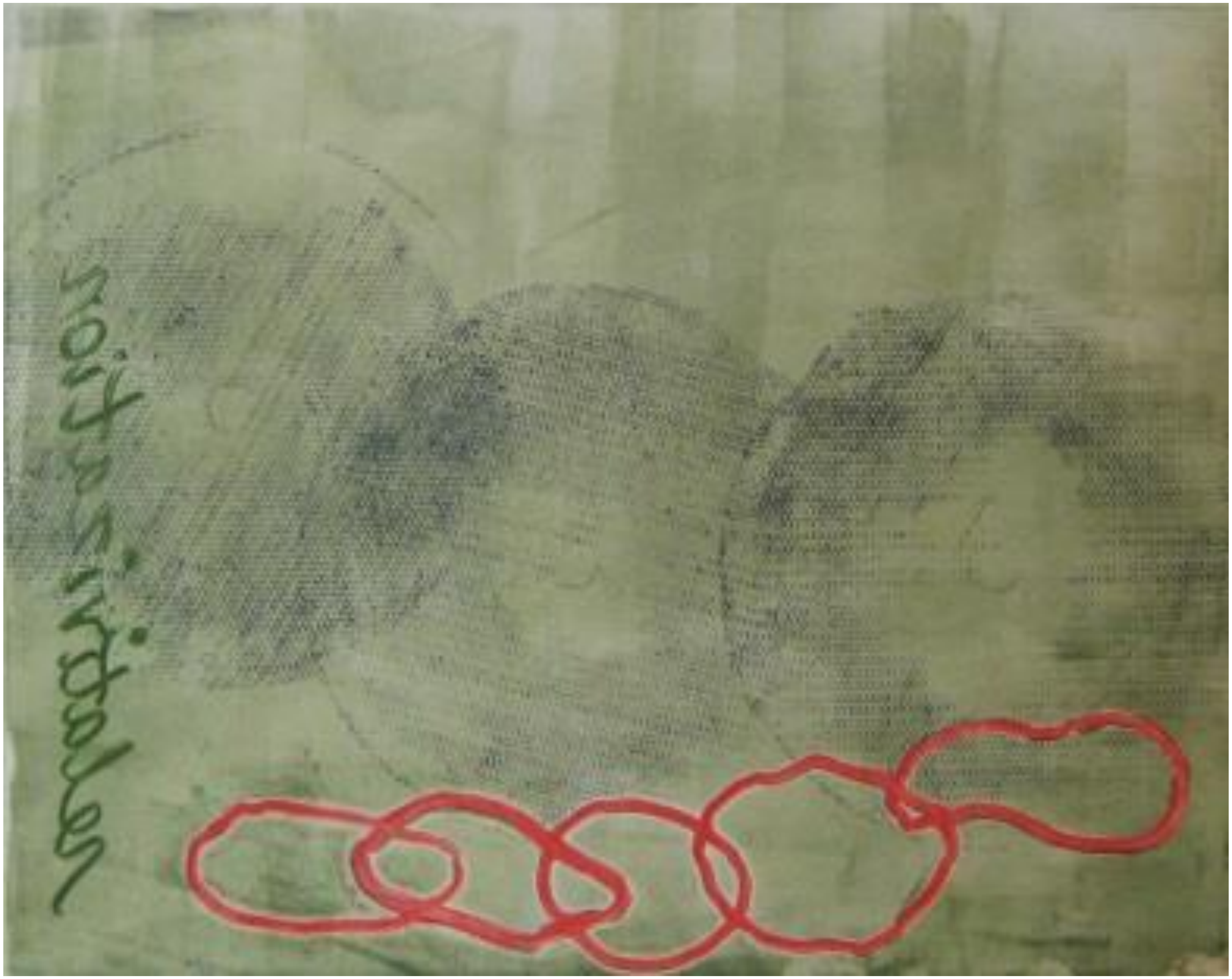
[17] Ehrenreich, 2003.

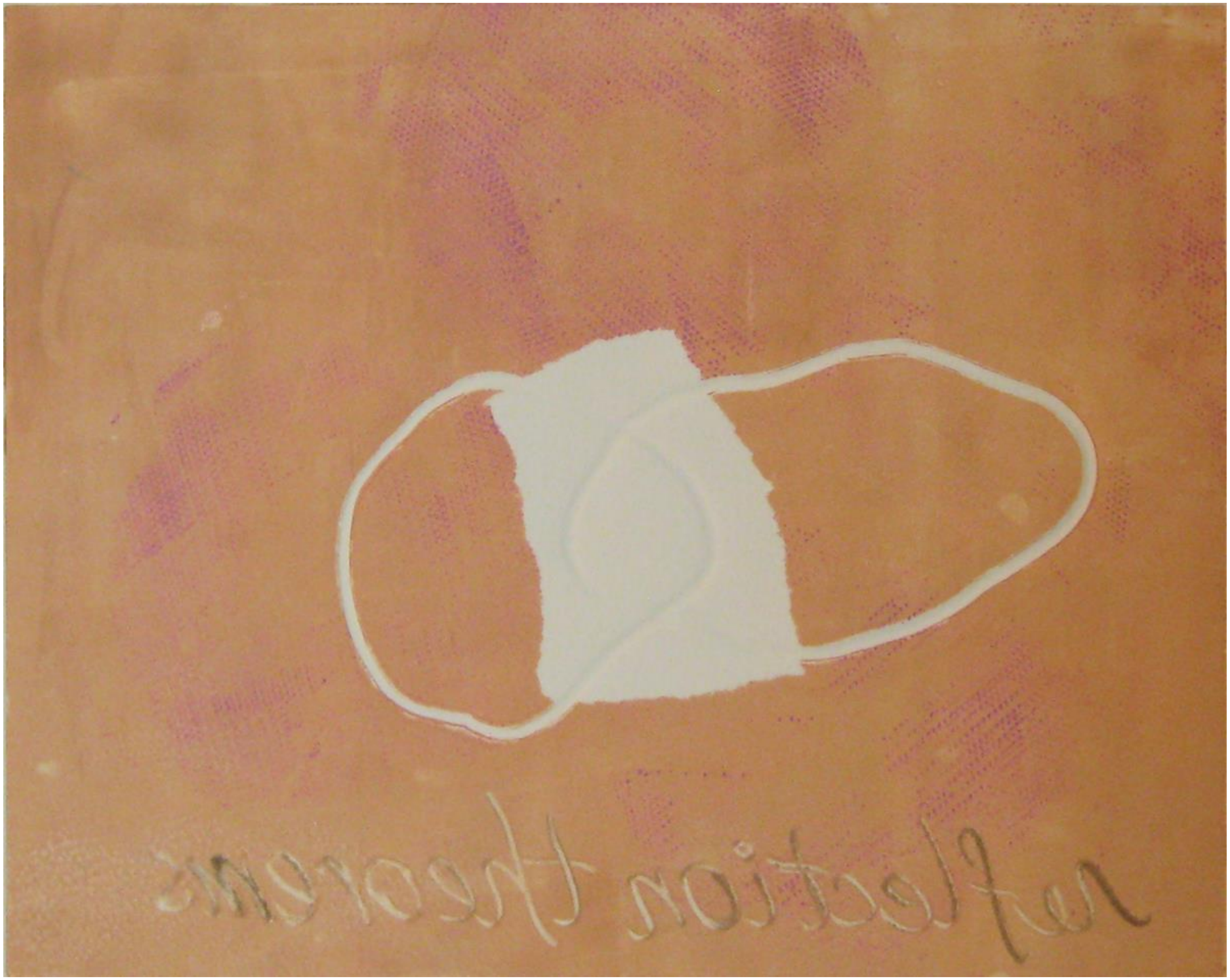
[18] Kunen, 1980.



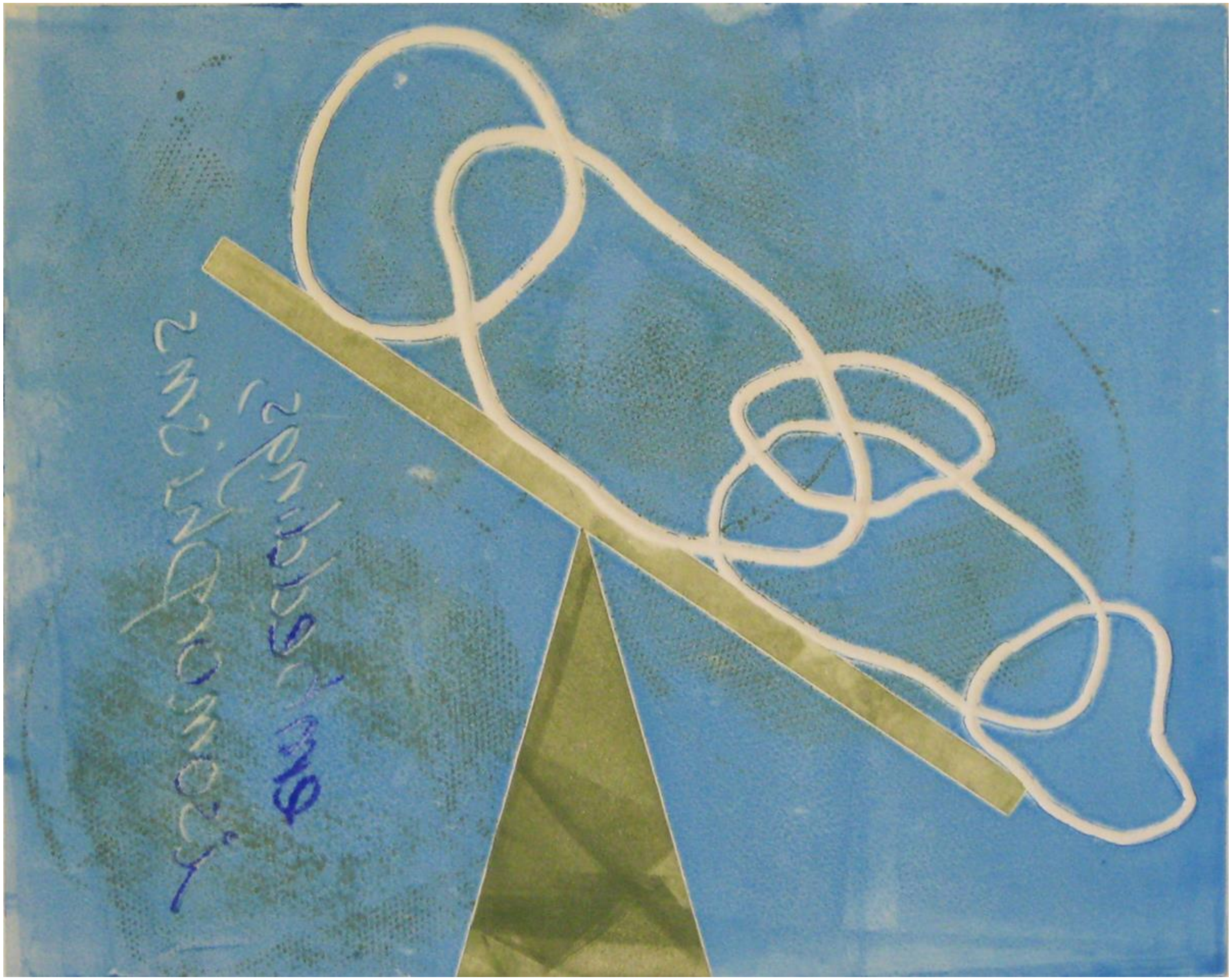
Exclusion, 11”x14”, ink on paper, 2010



Battle of Oppressions, 11”x14”, ink on paper, 2010

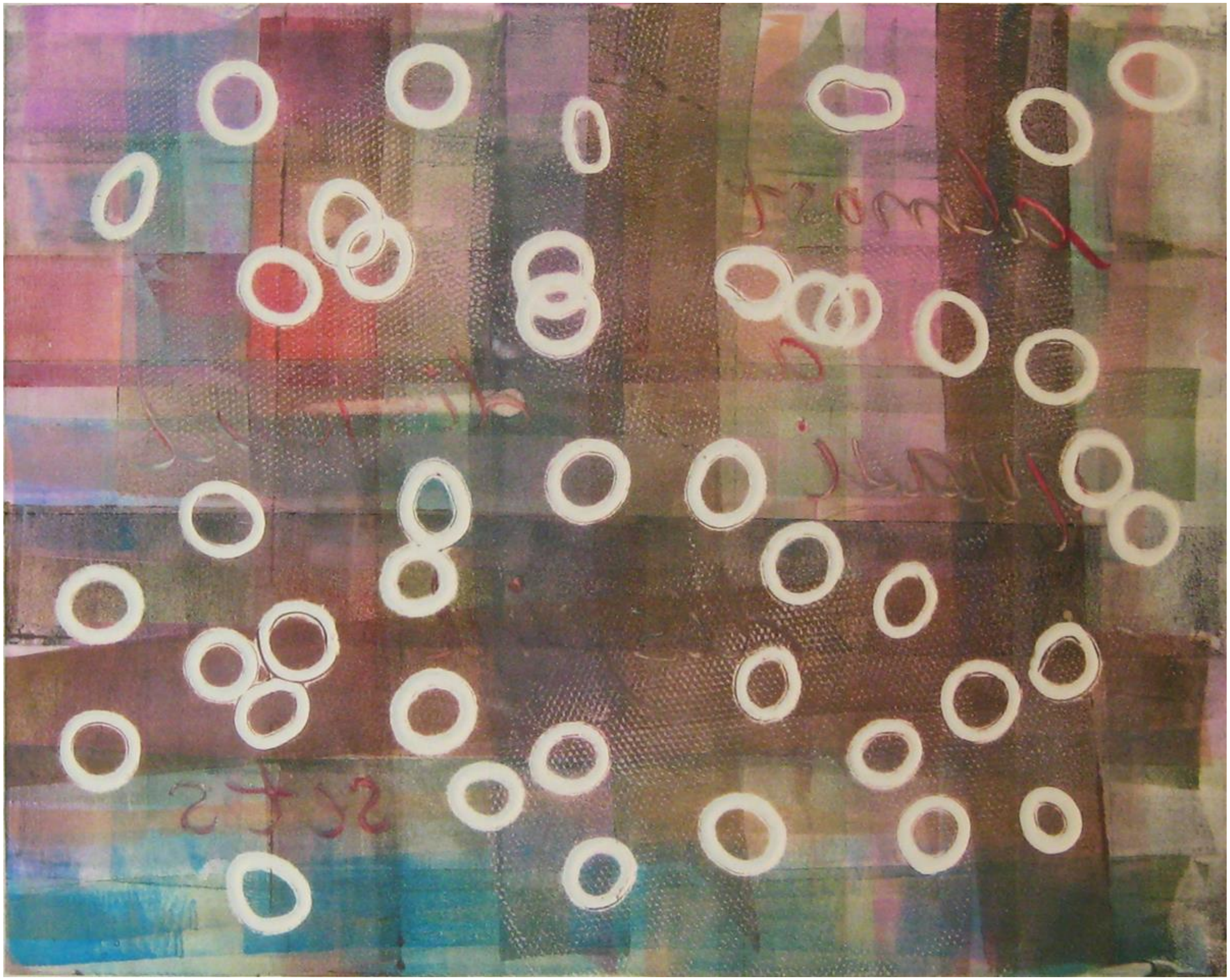


Reinforced Subordination, 11”x14”, ink on paper, 2010



Zero-Sum Problem, 11”x14”, ink on paper, 2010

Divide and Conquer, 11”x14”, ink on paper, 2010



Infinite Regress, 11”x14”, ink on paper, 2010

EQUATION MONOPRINTS

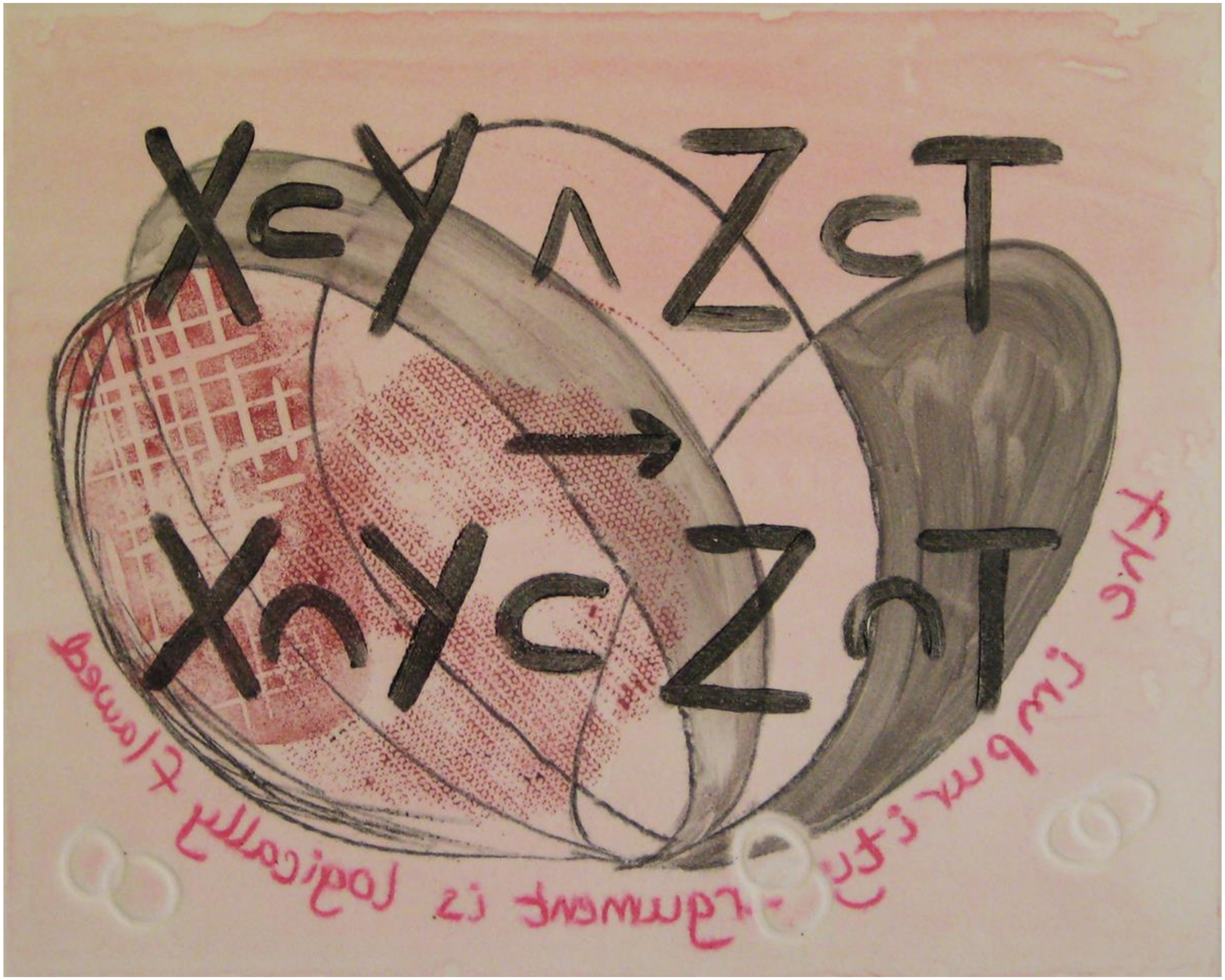
Each of these uniquely made prints illustrates its title: an equation or concept from a set-theory textbook[19]. The limited red and black palette recalls single-colour printing, once common in math textbooks.

Each mathematical concept is mirrored by a related phrase from Ehrenreich[20].

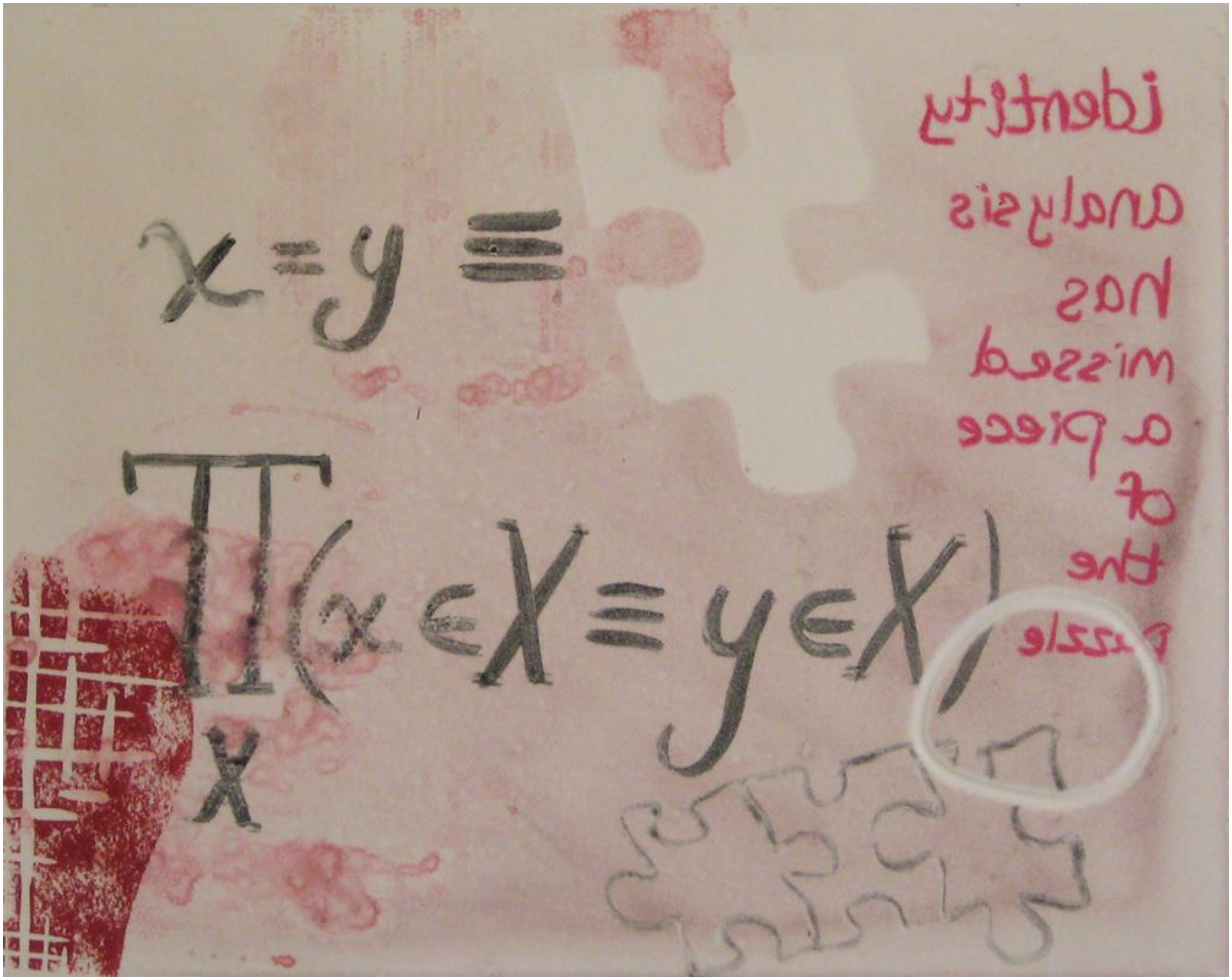
Print title, from Słupecki & Borkowski	**Ehrenreich quotation**
False formula	the impurity argument is logically flawed
Definition of the identity of individuals	Identity analysis has missed a piece of the puzzle
Infinite operations	Vectors of group identification
Set and complement	Exclusion works against the interests of the very group doing it… Rendering invisible some members of that identity group
Every set of individuals is a subset of the universal set	the proliferation of identity critiques
Relative product	two subordinated statuses are interacting

[19] Słupecki, J. and L. Borkowski, *Elements of Mathematical Logic and Set Theory* (Pergamon Press (Polish Scientific Publishers), 1967).

[20] Ehrenreich, 2003.

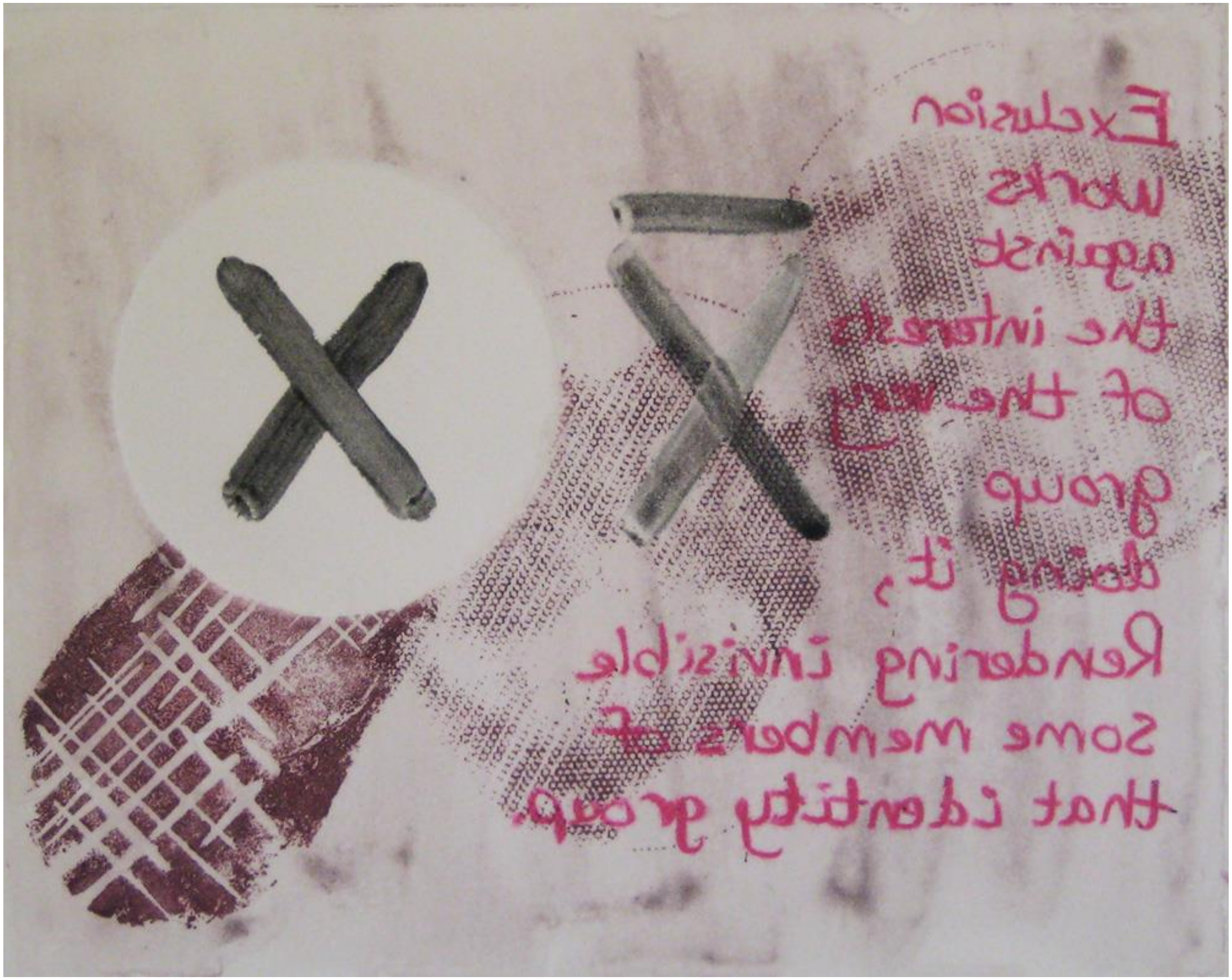


False Formula, 11”x14”, ink on paper, 2012

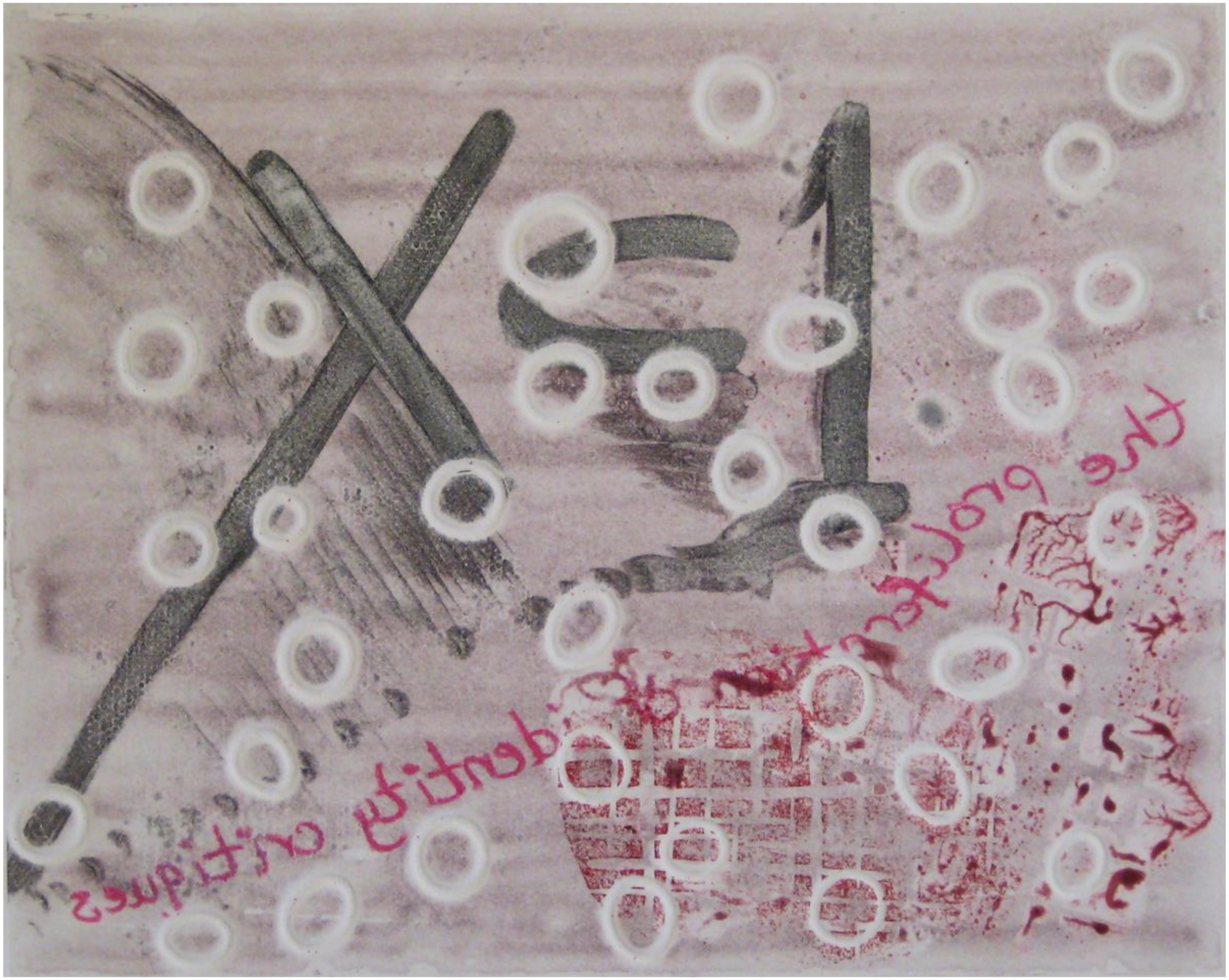


Definition of the identity of individuals, 11”x14”, ink on paper, 2012

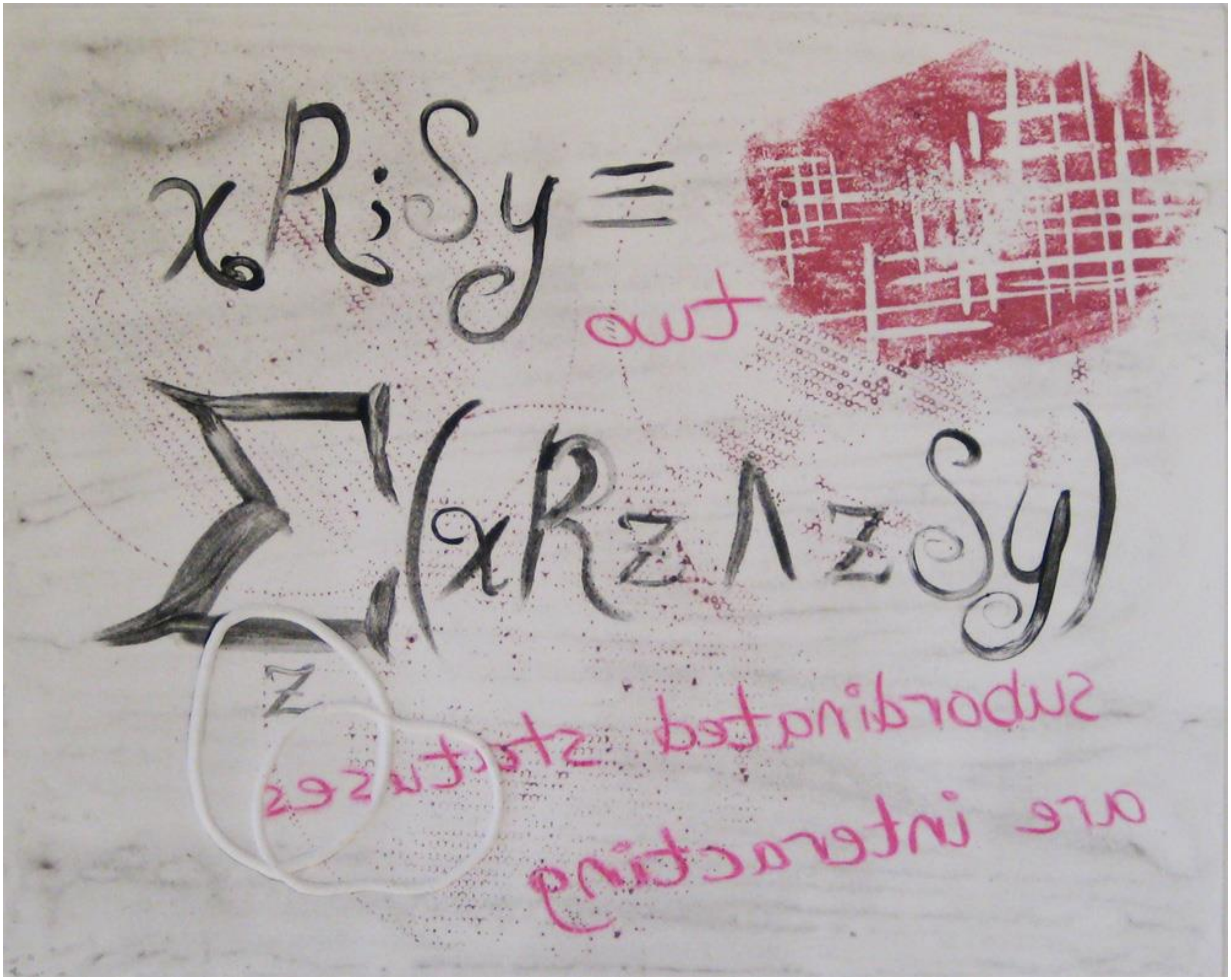
Infinite Operations, 11”x14”, ink on paper, 2012



Set and Complement, 11"x14", ink on paper, 2012



Every set of individuals is a subset of the universal set, 11”x14”, ink on paper, 2012



Relative Product, 11”x14”, ink on paper, 2012

Mathematical Logic in Catholic North America

To respond to the language of mathematics, I altered a book of conference papers[21] to become *Mathematical Logic in Catholic North America.* This mid-life crisis project was done in residency at Artscape Gibraltar Point, isolated in winter on the Toronto Islands.

The unrenovated school building, and the book's typography, brought back memories of growing up in the 1970s and 1980s in an intellectual family. (My father Paul Boltwood contributed the Dymo tape labels on the cover.)

On the book's pages I have written and drawn my distraught and delighted feelings about math, statistics, information management, class, gender, sexual orientation, conformity, religion, and truth.

The term "intersection entity" is used in the database design field, so I have drawn some entity-relationship models, such as page 33 (shown below). Other references include a mention of polynormativity[22] and collaged excerpts of art theory[23].

The milk crate book shelf evokes the graduate student life I never led.

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[21] Arruda, A.I. et al, editors, *Mathematical Logic in Latin America,* proceedings of the 4th Latin-American Symposium on Mathematical Logic, 1978 (North Holland Publishing, 1980).

[22] Zanin, Andrea. "The problem with polynormativity", posted 2013-01-24 on *Sex Geek: thoughts on sex and life* (blog), accessed March 30, 2014. http://sexgeek.wordpress.com/2013/01/24/theproblemwithpolynormativity/

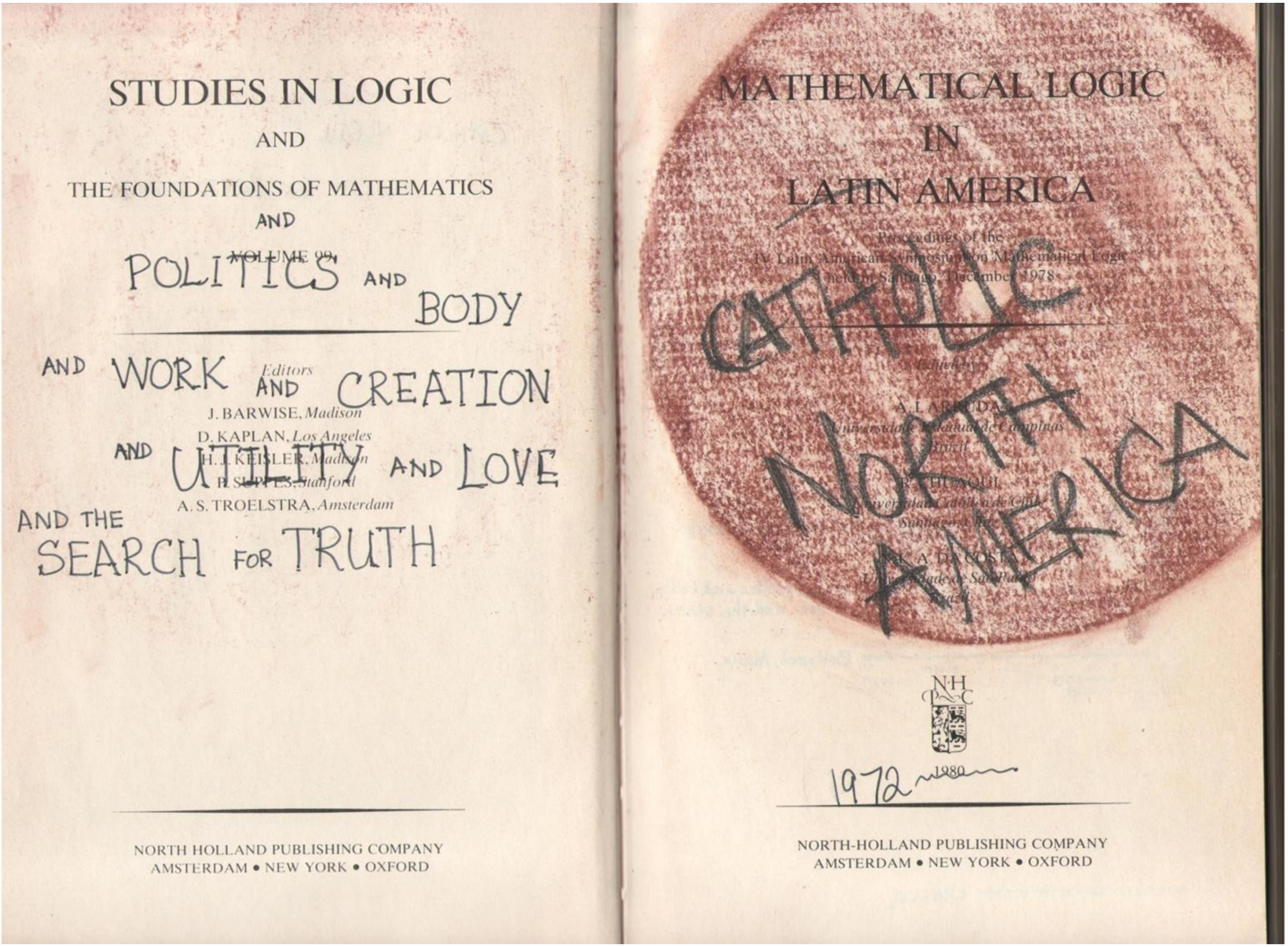
[23] Foster, 1987.

Mathematical Logic in Catholic North America, altered book, 2013.

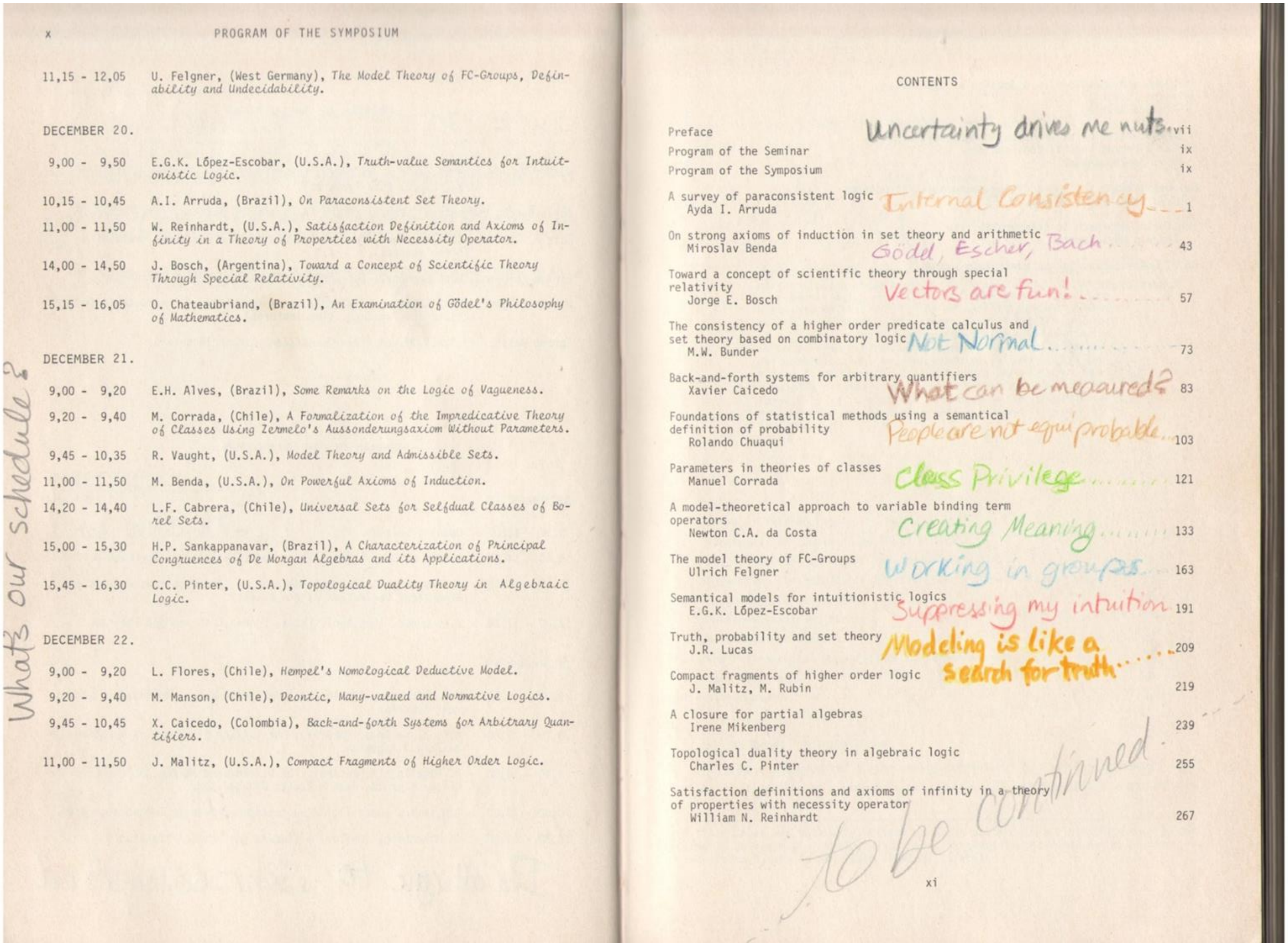
228 pages altered with marker and watercolour pencil, and Dymo tape on the hard cover.

Installed in a 13"x13"x11" milk crate with related books and documents.

Images of selected pages follow.



Mathematical Logic in Catholic North America, altered book, 2013, cover pages.



x PROGRAM OF THE SYMPOSIUM

11,15 - 12,05	U. Felgner, (West Germany), *The Model Theory of FC-Groups, Definability and Undecidability.*

DECEMBER 20.

9,00 - 9,50	E.G.K. López-Escobar, (U.S.A.), *Truth-value Semantics for Intuitionistic Logic.*
10,15 - 10,45	A.I. Arruda, (Brazil), *On Paraconsistent Set Theory.*
11,00 - 11,50	W. Reinhardt, (U.S.A.), *Satisfaction Definition and Axioms of Infinity in a Theory of Properties with Necessity Operator.*
14,00 - 14,50	J. Bosch, (Argentina), *Toward a Concept of Scientific Theory Through Special Relativity.*
15,15 - 16,05	O. Chateaubriand, (Brazil), *An Examination of Gödel's Philosophy of Mathematics.*

DECEMBER 21.

9,00 - 9,20	E.H. Alves, (Brazil), *Some Remarks on the Logic of Vagueness.*
9,20 - 9,40	M. Corrada, (Chile), *A Formalization of the Impredicative Theory of Classes Using Zermelo's Aussonderungsaxiom Without Parameters.*
9,45 - 10,35	R. Vaught, (U.S.A.), *Model Theory and Admissible Sets.*
11,00 - 11,50	M. Benda, (U.S.A.), *On Powerful Axioms of Induction.*
14,20 - 14,40	L.F. Cabrera, (Chile), *Universal Sets for Selfdual Classes of Borel Sets.*
15,00 - 15,30	H.P. Sankappanavar, (Brazil), *A Characterization of Principal Congruences of De Morgan Algebras and its Applications.*
15,45 - 16,30	C.C. Pinter, (U.S.A.), *Topological Duality Theory in Algebraic Logic.*

DECEMBER 22.

9,00 - 9,20	L. Flores, (Chile), *Hempel's Nomological Deductive Model.*
9,20 - 9,40	M. Manson, (Chile), *Deontic, Many-valued and Normative Logics.*
9,45 - 10,45	X. Caicedo, (Colombia), *Back-and-forth Systems for Arbitrary Quantifiers.*
11,00 - 11,50	J. Malitz, (U.S.A.), *Compact Fragments of Higher Order Logic.*

What's our schedule?

CONTENTS

to be continued.

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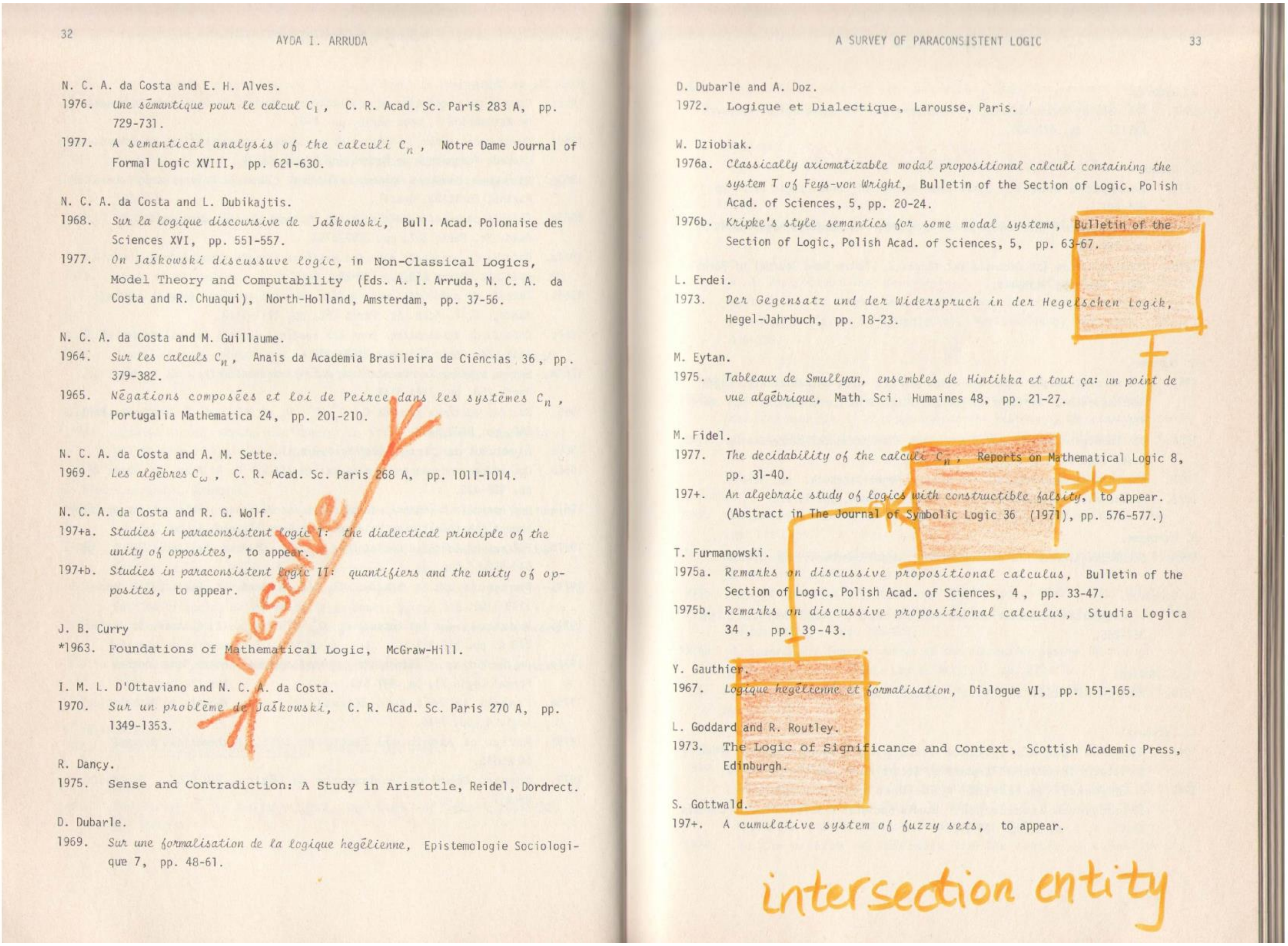


32 AYDA I. ARRUDA

N. C. A. da Costa and E. H. Alves.
1976. *Une sémantique pour le calcul C_1*, C. R. Acad. Sc. Paris 283 A, pp. 729-731.
1977. *A semantical analysis of the calculi C_n*, Notre Dame Journal of Formal Logic XVIII, pp. 621-630.

N. C. A. da Costa and L. Dubikajtis.
1968. *Sur la logique discoursive de Jaśkowski*, Bull. Acad. Polonaise des Sciences XVI, pp. 551-557.
1977. *On Jaśkowski discussuve logic*, in Non-Classical Logics, Model Theory and Computability (Eds. A. I. Arruda, N. C. A. da Costa and R. Chuaqui), North-Holland, Amsterdam, pp. 37-56.

N. C. A. da Costa and M. Guillaume.
1964. *Sur les calculs C_n*, Anais da Academia Brasileira de Ciências 36, pp. 379-382.
1965. *Négations composées et loi de Peirce dans les systèmes C_n*, Portugalia Mathematica 24, pp. 201-210.

N. C. A. da Costa and A. M. Sette.
1969. *Les algèbres C_ω*, C. R. Acad. Sc. Paris 268 A, pp. 1011-1014.

N. C. A. da Costa and R. G. Wolf.
197+a. *Studies in paraconsistent logic I: the dialectical principle of the unity of opposites*, to appear.
197+b. *Studies in paraconsistent logic II: quantifiers and the unity of opposites*, to appear.

J. B. Curry
*1963. Foundations of Mathematical Logic, McGraw-Hill.

I. M. L. D'Ottaviano and N. C. A. da Costa.
1970. *Sur un problème de Jaśkowski*, C. R. Acad. Sc. Paris 270 A, pp. 1349-1353.

R. Dançy.
1975. Sense and Contradiction: A Study in Aristotle, Reidel, Dordrect.

D. Dubarle.
1969. *Sur une formalisation de la logique hegélienne*, Epistemologie Sociologique 7, pp. 48-61.

A SURVEY OF PARACONSISTENT LOGIC 33

D. Dubarle and A. Doz.
1972. Logique et Dialectique, Larousse, Paris.

W. Dziobiak.
1976a. *Classically axiomatizable modal propositional calculi containing the system T of Feys-von Wright*, Bulletin of the Section of Logic, Polish Acad. of Sciences, 5, pp. 20-24.
1976b. *Kripke's style semantics for some modal systems*, Bulletin of the Section of Logic, Polish Acad. of Sciences, 5, pp. 63-67.

L. Erdei.
1973. *Der Gegensatz und der Widerspruch in der Hegelschen Logik*, Hegel-Jahrbuch, pp. 18-23.

M. Eytan.
1975. *Tableaux de Smullyan, ensembles de Hintikka et tout ça: un point de vue algébrique*, Math. Sci. Humaines 48, pp. 21-27.

M. Fidel.
1977. *The decidability of the calculi C_n*, Reports on Mathematical Logic 8, pp. 31-40.
197+. *An algebraic study of logics with constructible falsity*, to appear. (Abstract in The Journal of Symbolic Logic 36 (1971), pp. 576-577.)

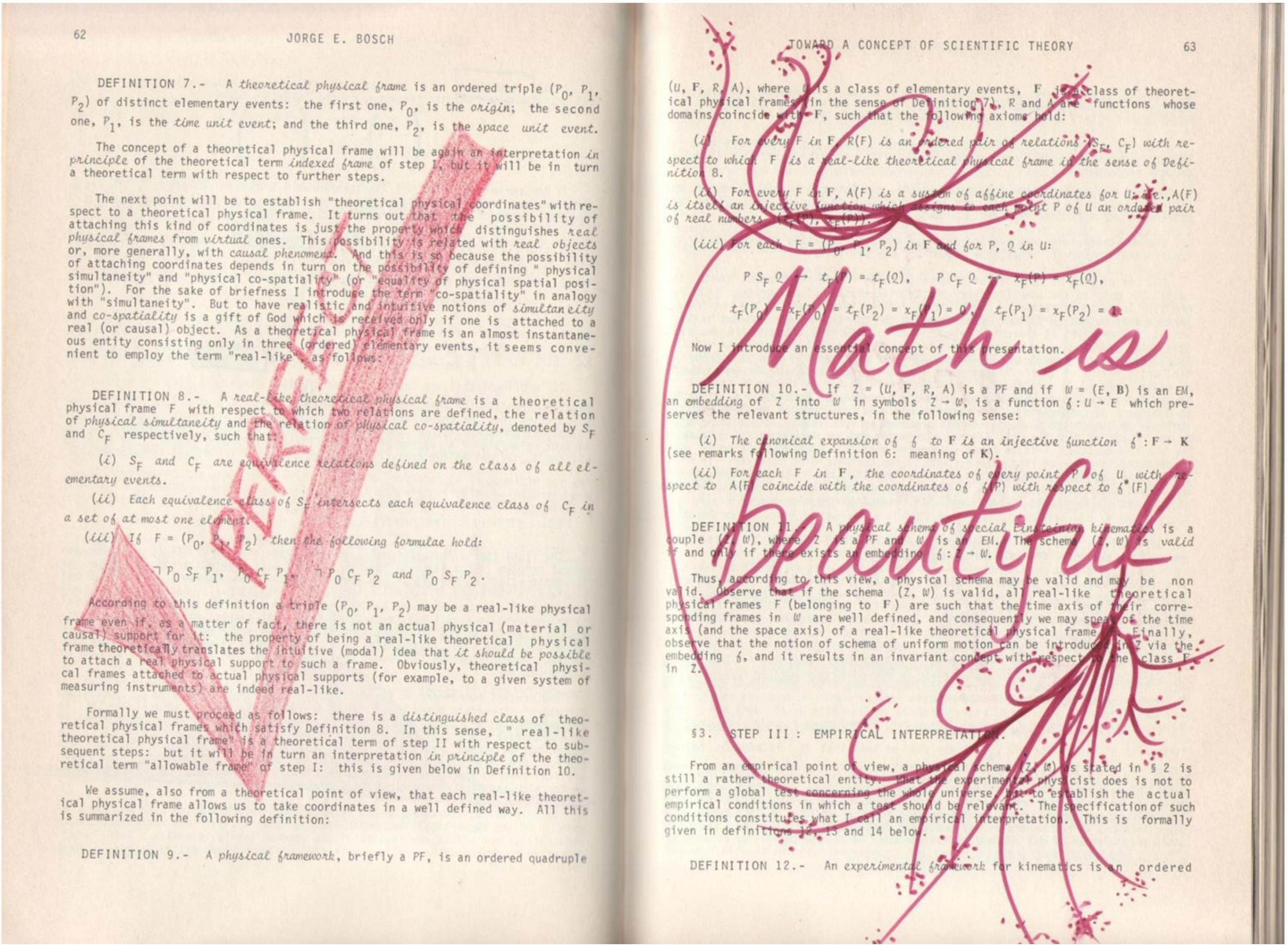
T. Furmanowski.
1975a. *Remarks on discussive propositional calculus*, Bulletin of the Section of Logic, Polish Acad. of Sciences, 4, pp. 33-47.
1975b. *Remarks on discussive propositional calculus*, Studia Logica 34, pp. 39-43.

Y. Gauthier.
1967. *Logique hegélienne et formalisation*, Dialogue VI, pp. 151-165.

L. Goddard and R. Routley.
1973. The Logic of Significance and Context, Scottish Academic Press, Edinburgh.

S. Gottwald.
197+. *A cumulative system of fuzzy sets*, to appear.

Mathematical Logic in Catholic North America, altered book, 2013, pp. 32-33.



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DEFINITION 7.- A *theoretical physical frame* is an ordered triple (P_0, P_1, P_2) of distinct elementary events: the first one, P_0, is the *origin*; the second one, P_1, is the *time unit event*; and the third one, P_2, is the *space unit event*.

The concept of a theoretical physical frame will be again an interpretation *in principle* of the theoretical term *indexed frame* of step I, but it will be in turn a theoretical term with respect to further steps.

The next point will be to establish "theoretical physical coordinates" with respect to a theoretical physical frame. It turns out that the possibility of attaching this kind of coordinates is just the property which distinguishes *real physical frames* from *virtual* ones. This possibility is related with *real objects* or, more generally, with *causal phenomena*. And this is so because the possibility of attaching coordinates depends in turn on the possibility of defining " physical simultaneity" and "physical co-spatiality" (or "equality of physical spatial position"). For the sake of briefness I introduce the term "co-spatiality" in analogy with "simultaneity". But to have realistic and intuitive notions of *simultaneity* and *co-spatiality* is a gift of God which is received only if one is attached to a real (or causal) object. As a theoretical physical frame is an almost instantaneous entity consisting only in three (ordered) elementary events, it seems convenient to employ the term "real-like", as follows:

DEFINITION 8.- A *real-like theoretical physical frame* is a theoretical physical frame F with respect to which two relations are defined, the relation of *physical simultaneity* and the relation of *physical co-spatiality*, denoted by S_F and C_F respectively, such that:

(*i*) *S_F and C_F are equivalence relations defined on the class of all elementary events.*

(*ii*) *Each equivalence class of S_F intersects each equivalence class of C_F in a set of at most one element.*

(*iii*) *If $F = (P_0, P_1, P_2)$ then the following formulae hold:*

$$\neg P_0 S_F P_1, \quad P_0 C_F P_1, \quad \neg P_0 C_F P_2 \quad \text{and} \quad P_0 S_F P_2 .$$

According to this definition a triple (P_0, P_1, P_2) may be a real-like physical frame even if, as a matter of fact, there is not an actual physical (material or causal) support for it: the property of being a real-like theoretical physical frame theoretically translates the intuitive (modal) idea that *it should be possible* to attach a real physical support to such a frame. Obviously, theoretical physical frames attached to actual physical supports (for example, to a given system of measuring instruments) are indeed real-like.

Formally we must proceed as follows: there is a *distinguished class* of theoretical physical frames which satisfy Definition 8. In this sense, " real-like theoretical physical frame" is a theoretical term of step II with respect to subsequent steps: but it will be in turn an interpretation *in principle* of the theoretical term "allowable frame" of step I: this is given below in Definition 10.

We assume, also from a theoretical point of view, that each real-like theoretical physical frame allows us to take coordinates in a well defined way. All this is summarized in the following definition:

DEFINITION 9.- A *physical framework*, briefly a *PF*, is an ordered quadruple

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$(U, \mathbf{F}, R, A)$, where U is a class of elementary events, $\mathbf{F}$ is a class of theoretical physical frames (in the sense of Definition 7), R and A are functions whose domains coincide with $\mathbf{F}$, such that the following axioms hold:

(*i*) *For every F in $\mathbf{F}$, R(F) is an ordered pair of relations (S_F, C_F) with respect to which F is a real-like theoretical physical frame in the sense of Definition 8.*

(*ii*) *For every F in $\mathbf{F}$, A(F) is a system of affine coordinates for U; i.e., A(F) is itself an injective function which assigns to each point P of U an ordered pair of real numbers $(t_F(P), x_F(P))$.*

(*iii*) *For each $F = (P_0, P_1, P_2)$ in $\mathbf{F}$ and for P, Q in U:*

$$P S_F Q \leftrightarrow t_F(P) = t_F(Q), \qquad P C_F Q \leftrightarrow x_F(P) = x_F(Q),$$

$$t_F(P_0) = x_F(P_0) = t_F(P_2) = x_F(P_1) = 0, \quad t_F(P_1) = x_F(P_2) = 1.$$

Now I introduce an essential concept of this presentation.

DEFINITION 10.- If $Z = (U, \mathbf{F}, R, A)$ is a *PF* and if $W = (E, \mathbf{B})$ is an *EM*, an *embedding* of Z into W in symbols $Z \to W$, is a function $f : U \to E$ which preserves the relevant structures, in the following sense:

(*i*) *The canonical expansion of f to $\mathbf{F}$ is an injective function $f^* : \mathbf{F} \to \mathbf{K}$* (see remarks following Definition 6: meaning of $\mathbf{K}$).

(*ii*) *For each F in $\mathbf{F}$, the coordinates of every point P of U with respect to* A(F) *coincide with the coordinates of* $f(P)$ *with respect to* $f^*(F)$.

DEFINITION 11.- A *physical schema of special Einsteinian kinematics* is a couple (Z, W), where Z is a *PF* and W is an *EM*. The schema (Z, W) is *valid* if and only if there exists an embedding $f : Z \to W$.

Thus, according to this view, a physical schema may be valid and may be non valid. Observe that if the schema (Z, W) is valid, all real-like theoretical physical frames F (belonging to $\mathbf{F}$) are such that the time axis of their corresponding frames in W are well defined, and consequently we may speak of the time axis (and the space axis) of a real-like theoretical physical frame F. Finally, observe that the notion of schema of uniform motion can be introduced in Z via the embedding f, and it results in an invariant concept with respect to the class $\mathbf{F}$ in Z.

§3. STEP III : EMPIRICAL INTERPRETATION.

From an empirical point of view, a physical schema (Z, W) as stated in § 2 is still a rather theoretical entity. What the experimental physicist does is not to perform a global test concerning the whole universe, but to establish the actual empirical conditions in which a test should be relevant. The specification of such conditions constitutes what I call an empirical interpretation. This is formally given in definitions 12, 13 and 14 below.

DEFINITION 12.- An *experimental framework* for kinematics is an ordered

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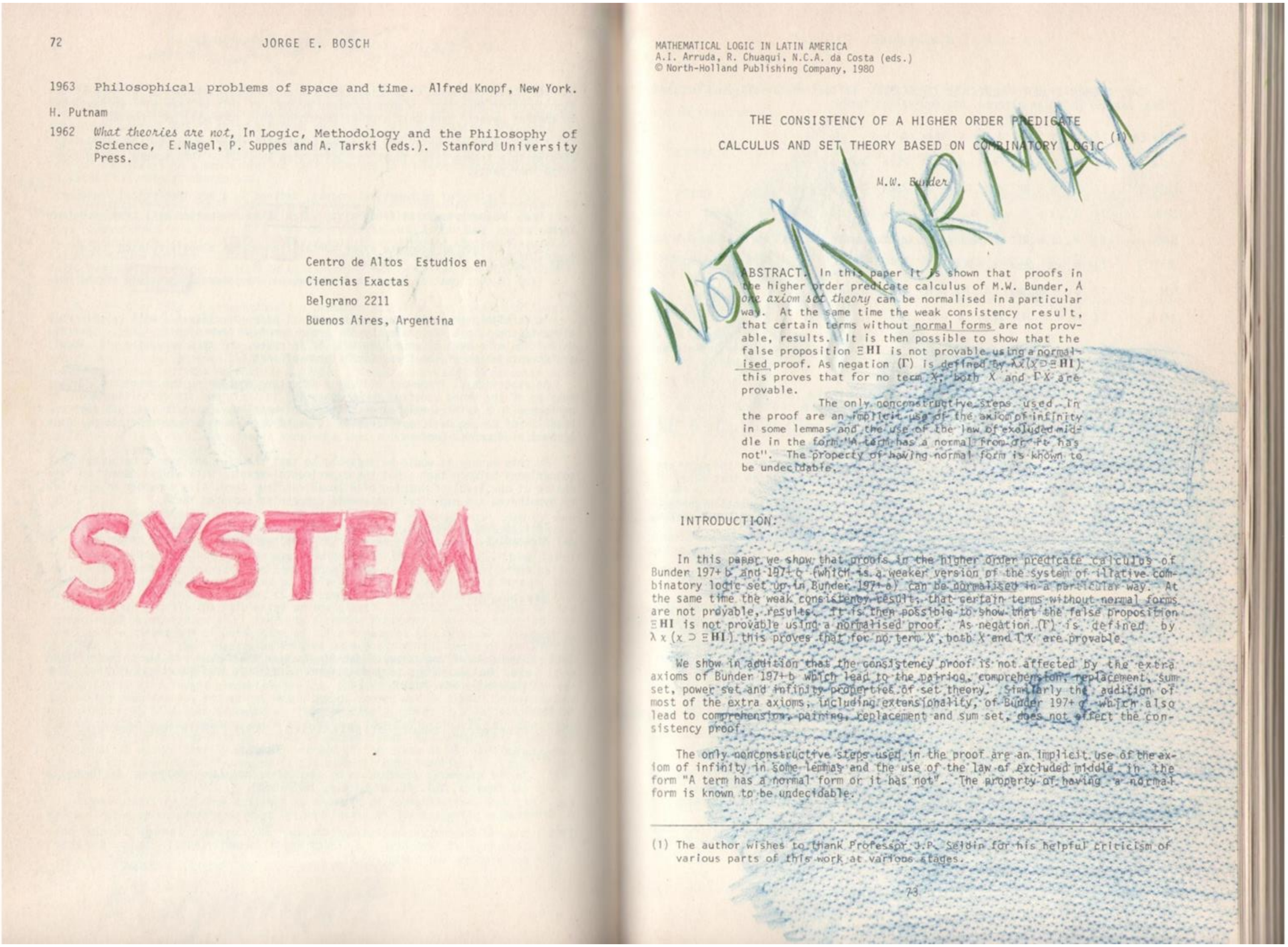

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SYSTEM

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THE CONSISTENCY OF A HIGHER ORDER PREDICATE
CALCULUS AND SET THEORY BASED ON COMBINATORY LOGIC (1)

M.W. Bunder

NOT NORMAL

ABSTRACT. In this paper it is shown that proofs in the higher order predicate calculus of M.W. Bunder, *A one axiom set theory* can be normalised in a particular way. At the same time the weak consistency result, that certain terms without normal forms are not provable, results. It is then possible to show that the false proposition $\Xi\mathbf{HI}$ is not provable using a normalised proof. As negation (Γ) is defined by $\lambda x(x \supset \Xi\mathbf{HI})$ this proves that for no term X, both X and ΓX are provable.

The only nonconstructive steps used in the proof are an implicit use of the axiom of infinity in some lemmas and the use of the law of excluded middle in the form "A term has a normal from or it has not". The property of having normal form is known to be undecidable.

INTRODUCTION:

In this paper we show that proofs in the higher order predicate calculus of Bunder 197+ b and 197+ c (which is a weaker version of the system of illative combinatory logic set up in Bunder 197+ a) can be normalised in a particular way. At the same time the weak consistency result, that certain terms without normal forms are not provable, results. It is then possible to show that the false proposition $\Xi\mathbf{HI}$ is not provable using a normalised proof. As negation (Γ) is defined by $\lambda x(x \supset \Xi\mathbf{HI})$ this proves that for no term X, both X and ΓX are provable.

We show in addition that the consistency proof is not affected by the extra axioms of Bunder 197+ b which lead to the pairing, comprehension, replacement, sum set, power set and infinity properties of set theory. Similarly the addition of most of the extra axioms, including extensionality, of Bunder 197+ c which also lead to comprehension, pairing, replacement and sum set, does not effect the consistency proof.

The only nonconstructive steps used in the proof are an implicit use of the axiom of infinity in some lemmas and the use of the law of excluded middle in the form "A term has a normal form or it has not". The property of having a normal form is known to be undecidable.

(1) The author wishes to thank Professor J.P. Seldin for his helpful criticism of various parts of this work at various stages.

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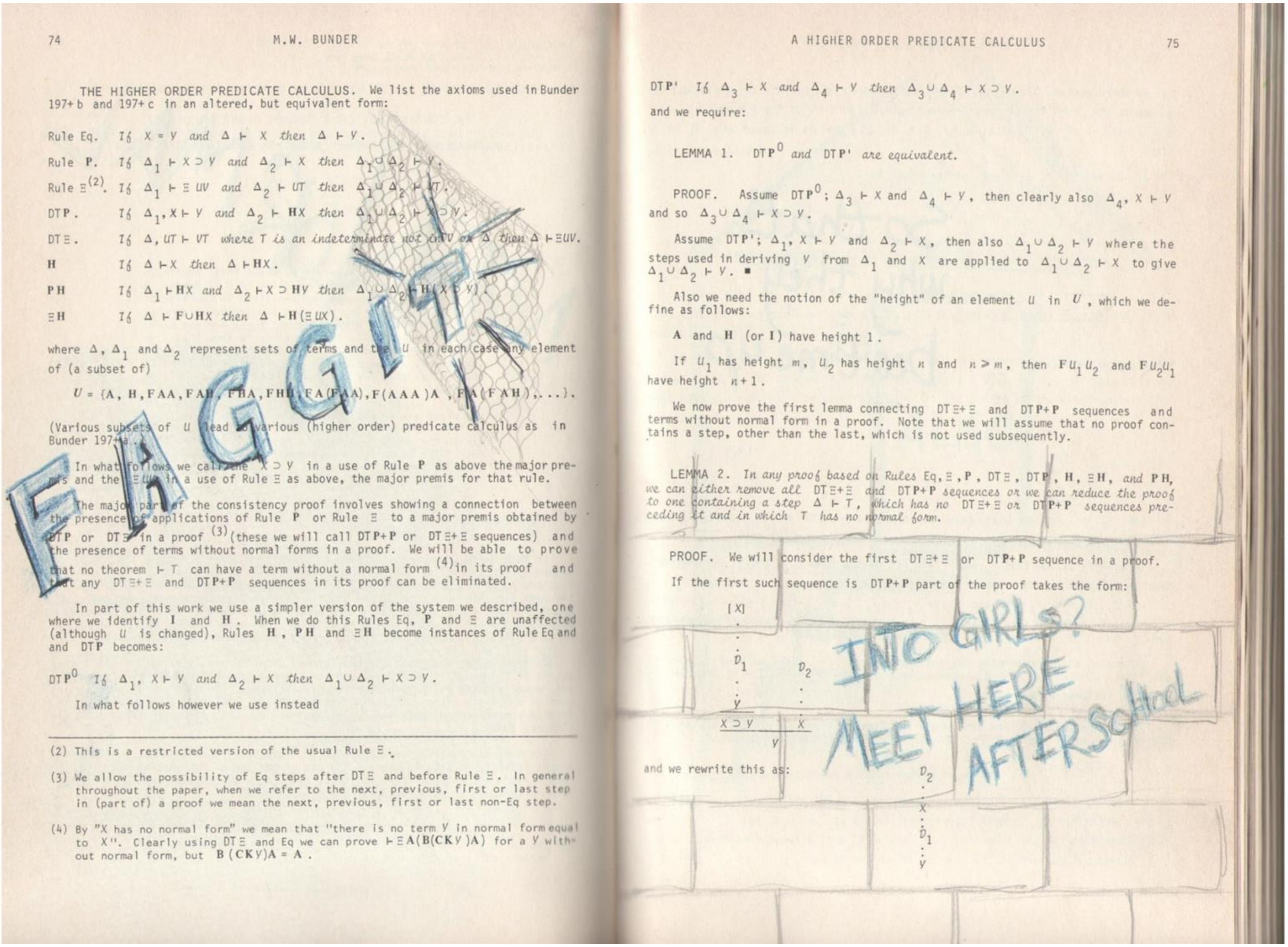
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THE HIGHER ORDER PREDICATE CALCULUS. We list the axioms used in Bunder 197+b and 197+c in an altered, but equivalent form:

Rule Eq. *If* $X = Y$ *and* $\Delta \vdash X$ *then* $\Delta \vdash Y$.

Rule **P**. *If* $\Delta_1 \vdash X \supset Y$ *and* $\Delta_2 \vdash X$ *then* $\Delta_1 \cup \Delta_2 \vdash Y$.

Rule $\Xi^{(2)}$. *If* $\Delta_1 \vdash \Xi UV$ *and* $\Delta_2 \vdash UT$ *then* $\Delta_1 \cup \Delta_2 \vdash VT$.

DT**P**. *If* $\Delta_1, X \vdash Y$ *and* $\Delta_2 \vdash \mathbf{H}X$ *then* $\Delta_1 \cup \Delta_2 \vdash X \supset Y$.

DTΞ. *If* $\Delta, UT \vdash VT$ *where* T *is an indeterminate not in* U *or* Δ *then* $\Delta \vdash \Xi UV$.

H *If* $\Delta \vdash X$ *then* $\Delta \vdash \mathbf{H}X$.

PH *If* $\Delta_1 \vdash \mathbf{H}X$ *and* $\Delta_2 \vdash X \supset \mathbf{H}Y$ *then* $\Delta_1 \cup \Delta_2 \vdash \mathbf{H}(X \supset Y)$.

Ξ**H** *If* $\Delta \vdash \mathbf{F}U\mathbf{H}X$ *then* $\Delta \vdash \mathbf{H}(\Xi UX)$.

where Δ, Δ_1 and Δ_2 represent sets of terms and the U in each case any element of (a subset of)

$$U = \{\mathbf{A}, \mathbf{H}, \mathbf{FAA}, \mathbf{FAH}, \mathbf{FHA}, \mathbf{FHH}, \mathbf{FA}(\mathbf{FAA}), \mathbf{F}(\mathbf{AAA})\mathbf{A}, \mathbf{FA}(\mathbf{FAH}), \ldots\}.$$

(Various subsets of U lead to various (higher order) predicate calculus as in Bunder 197+a.)

In what follows we call the $X \supset Y$ in a use of Rule **P** as above the major premis and the ΞUV in a use of Rule Ξ as above, the major premis for that rule.

The major part of the consistency proof involves showing a connection between the presence of applications of Rule **P** or Rule Ξ to a major premis obtained by DT**P** or DTΞ in a proof [3] (these we will call DT**P**+**P** or DTΞ+Ξ sequences) and the presence of terms without normal forms in a proof. We will be able to prove that no theorem $\vdash T$ can have a term without a normal form [4] in its proof and that any DTΞ+Ξ and DT**P**+**P** sequences in its proof can be eliminated.

In part of this work we use a simpler version of the system we described, one where we identify **I** and **H**. When we do this Rules Eq, **P** and Ξ are unaffected (although U is changed), Rules **H**, **PH** and Ξ**H** become instances of Rule Eq and and DT**P** becomes:

DT$\mathbf{P}^0$ *If* $\Delta_1, X \vdash Y$ *and* $\Delta_2 \vdash X$ *then* $\Delta_1 \cup \Delta_2 \vdash X \supset Y$.

In what follows however we use instead

(2) This is a restricted version of the usual Rule Ξ.

(3) We allow the possibility of Eq steps after DTΞ and before Rule Ξ. In general throughout the paper, when we refer to the next, previous, first or last step in (part of) a proof we mean the next, previous, first or last non-Eq step.

(4) By "X has no normal form" we mean that "there is no term Y in normal form equal to X". Clearly using DTΞ and Eq we can prove $\vdash \Xi\mathbf{A}(\mathbf{B}(\mathbf{CK}Y)\mathbf{A})$ for a Y without normal form, but $\mathbf{B}(\mathbf{CK}Y)\mathbf{A} = \mathbf{A}$.

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DT**P**' *If* $\Delta_3 \vdash X$ *and* $\Delta_4 \vdash Y$ *then* $\Delta_3 \cup \Delta_4 \vdash X \supset Y$.

and we require:

LEMMA 1. DT$\mathbf{P}^0$ *and* DT**P**' *are equivalent.*

PROOF. Assume DT$\mathbf{P}^0$; $\Delta_3 \vdash X$ and $\Delta_4 \vdash Y$, then clearly also $\Delta_4, X \vdash Y$ and so $\Delta_3 \cup \Delta_4 \vdash X \supset Y$.

Assume DT**P**'; $\Delta_1, X \vdash Y$ and $\Delta_2 \vdash X$, then also $\Delta_1 \cup \Delta_2 \vdash Y$ where the steps used in deriving Y from Δ_1 and X are applied to $\Delta_1 \cup \Delta_2 \vdash X$ to give $\Delta_1 \cup \Delta_2 \vdash Y$. ∎

Also we need the notion of the "height" of an element U in U, which we define as follows:

A and **H** (or **I**) have height 1.

If U_1 has height m, U_2 has height n and $n \geq m$, then $\mathbf{F}U_1U_2$ and $\mathbf{F}U_2U_1$ have height $n+1$.

We now prove the first lemma connecting DTΞ+Ξ and DT**P**+**P** sequences and terms without normal form in a proof. Note that we will assume that no proof contains a step, other than the last, which is not used subsequently.

LEMMA 2. *In any proof based on Rules* Eq, Ξ, **P**, DTΞ, DT**P**, **H**, Ξ**H**, *and* **PH**, *we can either remove all* DTΞ+Ξ *and* DT**P**+**P** *sequences or we can reduce the proof to one containing a step* $\Delta \vdash T$, *which has no* DTΞ+Ξ *or* DT**P**+**P** *sequences preceding it and in which* T *has no normal form.*

PROOF. We will consider the first DTΞ+Ξ or DT**P**+**P** sequence in a proof.

If the first such sequence is DT**P**+**P** part of the proof takes the form:

$$\dfrac{\dfrac{\begin{matrix}[X]\\ \vdots\\ \mathcal{D}_1\\ \vdots\\ Y\end{matrix}}{X \supset Y} \qquad \begin{matrix}\mathcal{D}_2\\ \vdots\\ X\end{matrix}}{Y}$$

and we rewrite this as:

$$\begin{matrix}\mathcal{D}_2\\ \vdots\\ X\\ \vdots\\ \mathcal{D}_1\\ \vdots\\ Y\end{matrix}$$

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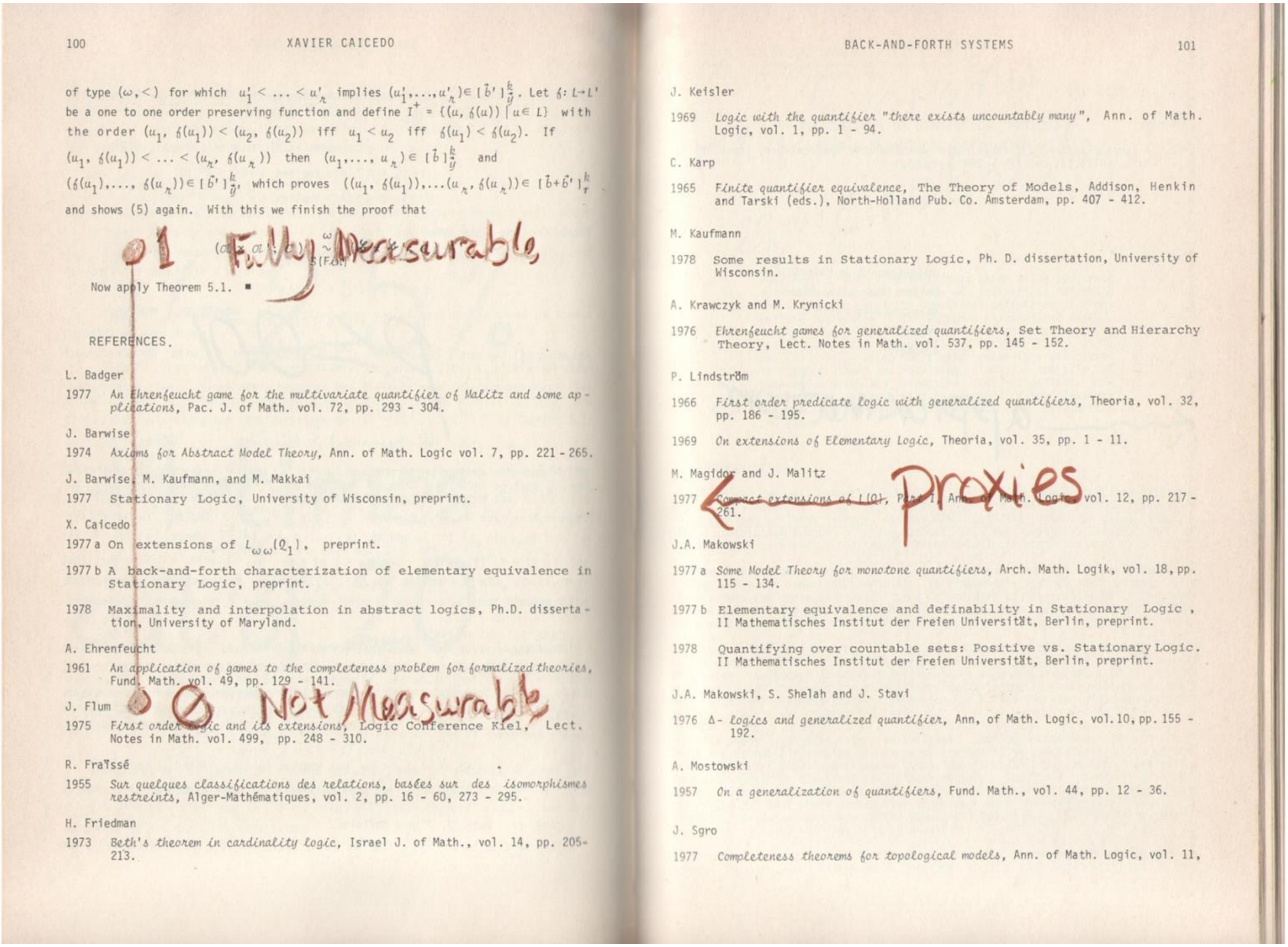
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of type $(\omega,<)$ for which $u'_1 < \ldots < u'_r$ implies $(u'_1,\ldots,u'_r) \in [\vec{b}']^k_y$. Let $f: L \to L'$ be a one to one order preserving function and define $I^+ = \{(u, f(u)) \mid u \in L\}$ with the order $(u_1, f(u_1)) < (u_2, f(u_2))$ iff $u_1 < u_2$ iff $f(u_1) < f(u_2)$. If $(u_1, f(u_1)) < \ldots < (u_r, f(u_r))$ then $(u_1,\ldots, u_r) \in [\vec{b}]^k_y$ and $(f(u_1),\ldots, f(u_r)) \in [\vec{b}']^k_y$, which proves $((u_1, f(u_1)),\ldots(u_r, f(u_r))) \in [\vec{b}+\vec{b}']^k_r$ and shows (5) again. With this we finish the proof that

1 Fully Measurable

Now apply Theorem 5.1. ■

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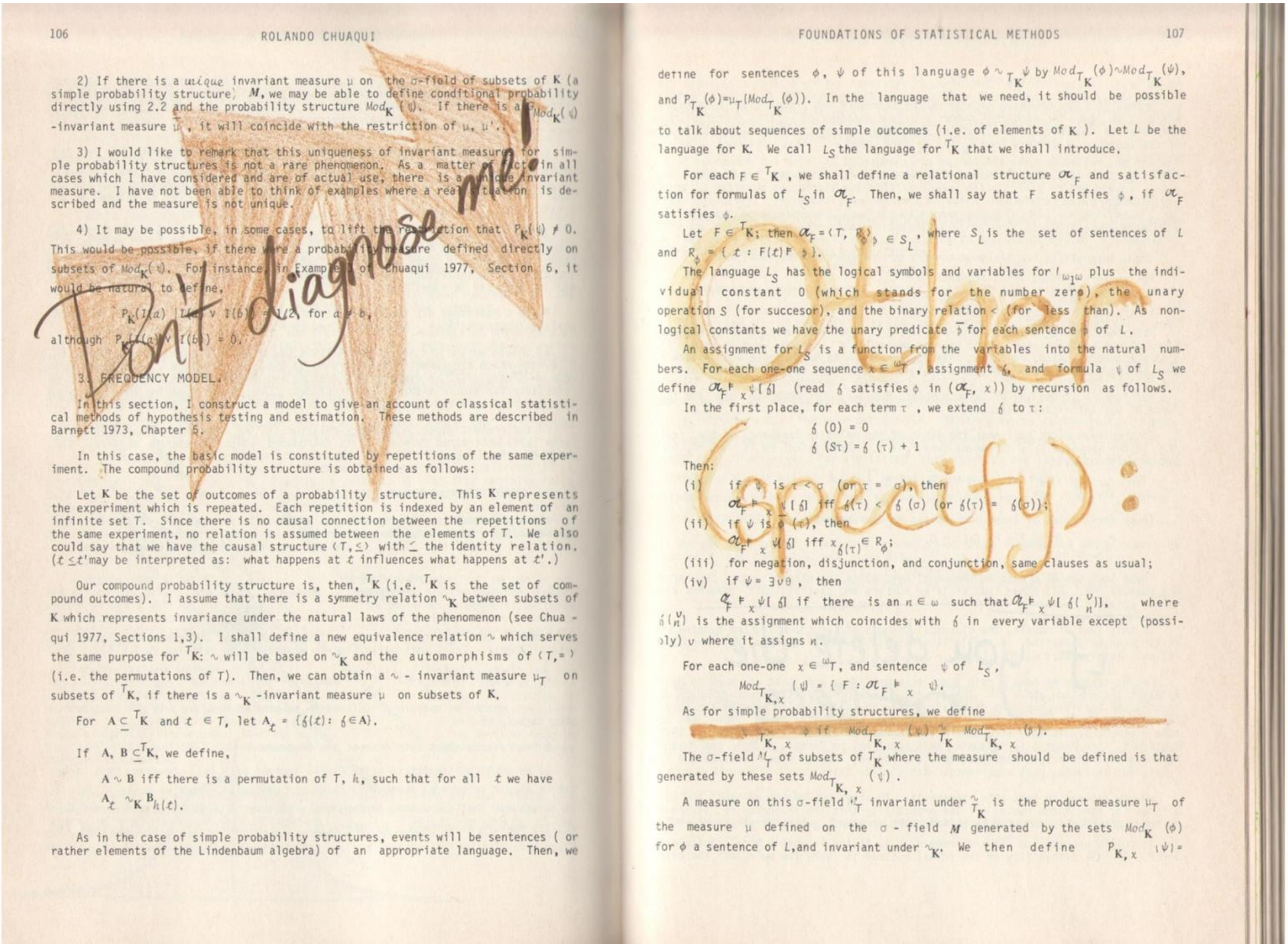

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2) If there is a *unique* invariant measure μ on the σ-field of subsets of $\mathbf{K}$ (a simple probability structure) M, we may be able to define conditional probability directly using 2.2 and the probability structure $Mod_{\mathbf{K}}(\psi)$. If there is a $G_{Mod_{\mathbf{K}}(\psi)}$-invariant measure $\bar{\mu}$, it will coincide with the restriction of μ, μ'..

3) I would like to remark that this uniqueness of invariant measures for simple probability structures is not a rare phenomenon. As a matter of fact, in all cases which I have considered and are of actual use, there is a unique invariant measure. I have not been able to think of examples where a real situation is described and the measure is not unique.

4) It may be possible, in some cases, to lift the restriction that $P_{\mathbf{K}}(\psi) \neq 0$. This would be possible, if there were a probability measure defined directly on subsets of $Mod_{\mathbf{K}}(\psi)$. For instance, in Example 3 of Chuaqui 1977, Section 6, it would be natural to define,

$$P_{\mathbf{K}}(I(a) \mid I(a) \vee I(b)) = 1/2, \text{ for } a \neq b,$$

although $P_{\mathbf{K}}(I(a) \vee I(b)) = 0$.

3. FREQUENCY MODEL.

In this section, I construct a model to give an account of classical statistical methods of hypothesis testing and estimation. These methods are described in Barnett 1973, Chapter 5.

In this case, the basic model is constituted by repetitions of the same experiment. The compound probability structure is obtained as follows:

Let $\mathbf{K}$ be the set of outcomes of a probability structure. This $\mathbf{K}$ represents the experiment which is repeated. Each repetition is indexed by an element of an infinite set T. Since there is no causal connection between the repetitions of the same experiment, no relation is assumed between the elements of T. We also could say that we have the causal structure $\langle T, \leq \rangle$ with $\leq$ the identity relation. ($t \leq t'$ may be interpreted as: what happens at t influences what happens at t'.)

Our compound probability structure is, then, ${}^{T}\mathbf{K}$ (i.e. ${}^{T}\mathbf{K}$ is the set of compound outcomes). I assume that there is a symmetry relation $\sim_{\mathbf{K}}$ between subsets of $\mathbf{K}$ which represents invariance under the natural laws of the phenomenon (see Chuaqui 1977, Sections 1,3). I shall define a new equivalence relation $\sim$ which serves the same purpose for ${}^{T}\mathbf{K}$: $\sim$ will be based on $\sim_{\mathbf{K}}$ and the automorphisms of $\langle T, = \rangle$ (i.e. the permutations of T). Then, we can obtain a $\sim$-invariant measure μ_T on subsets of ${}^{T}\mathbf{K}$, if there is a $\sim_{\mathbf{K}}$-invariant measure μ on subsets of $\mathbf{K}$.

For $\mathbf{A} \subseteq {}^{T}\mathbf{K}$ and $t \in T$, let $\mathbf{A}_t = \{f(t) : f \in \mathbf{A}\}$.

If $\mathbf{A}, \mathbf{B} \subseteq {}^{T}\mathbf{K}$, we define,

$\mathbf{A} \sim \mathbf{B}$ iff there is a permutation of T, h, such that for all t we have $\mathbf{A}_t \sim_{\mathbf{K}} \mathbf{B}_{h(t)}$.

As in the case of simple probability structures, events will be sentences (or rather elements of the Lindenbaum algebra) of an appropriate language. Then, we

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define for sentences ϕ, ψ of this language $\phi \sim_{{}^{T}\mathbf{K}} \psi$ by $Mod_{{}^{T}\mathbf{K}}(\phi) \sim Mod_{{}^{T}\mathbf{K}}(\psi)$, and $P_{{}^{T}\mathbf{K}}(\phi) = \mu_T(Mod_{{}^{T}\mathbf{K}}(\phi))$. In the language that we need, it should be possible to talk about sequences of simple outcomes (i.e. of elements of $\mathbf{K}$). Let L be the language for $\mathbf{K}$. We call L_S the language for ${}^{T}\mathbf{K}$ that we shall introduce.

For each $F \in {}^{T}\mathbf{K}$, we shall define a relational structure $\mathfrak{A}_F$ and satisfaction for formulas of L_S in $\mathfrak{A}_F$. Then, we shall say that F satisfies ϕ, if $\mathfrak{A}_F$ satisfies ϕ.

Let $F \in {}^{T}\mathbf{K}$; then $\mathfrak{A}_F = \langle T, R_\phi \rangle_{\phi \in S_L}$, where S_L is the set of sentences of L and $R_\phi = \{t : F(t) \models \phi\}$.

The language L_S has the logical symbols and variables for $L_{\omega_1\omega}$ plus the individual constant 0 (which stands for the number zero), the unary operation S (for succesor), and the binary relation $<$ (for less than). As non-logical constants we have the unary predicate $\bar{\phi}$ for each sentence ϕ of L.

An assignment for L_S is a function from the variables into the natural numbers. For each one-one sequence $x \in {}^{\omega}T$, assignment f, and formula ψ of L_S we define $\mathfrak{A}_F \models_x \psi[f]$ (read f satisfies ϕ in $(\mathfrak{A}_F, x)$) by recursion as follows.

In the first place, for each term τ, we extend f to τ:

$$f(0) = 0$$
$$f(S\tau) = f(\tau) + 1$$

Then:

(i) if ψ is $\tau < \sigma$ (or $\tau = \sigma$), then $\mathfrak{A}_F \models_x \psi[f]$ iff $f(\tau) < f(\sigma)$ (or $f(\tau) = f(\sigma)$);

(ii) if ψ is $\bar{\phi}(\tau)$, then $\mathfrak{A}_F \models_x \psi[f]$ iff $x_{f(\tau)} \in R_\phi$;

(iii) for negation, disjunction, and conjunction, same clauses as usual;

(iv) if $\psi = \exists v\theta$, then $\mathfrak{A}_F \models_x \psi[f]$ if there is an $n \in \omega$ such that $\mathfrak{A}_F \models_x \psi[f(\tfrac{v}{n})]$, where $f(\tfrac{v}{n})$ is the assignment which coincides with f in every variable except (possibly) v where it assigns n.

For each one-one $x \in {}^{\omega}T$, and sentence ψ of L_S,

$$Mod_{{}^{T}\mathbf{K},x}(\psi) = \{F : \mathfrak{A}_F \models_x \psi\}.$$

As for simple probability structures, we define

~~$\psi \sim_{{}^{T}\mathbf{K},x} \phi$ if $Mod_{{}^{T}\mathbf{K},x}(\psi) \sim_{{}^{T}\mathbf{K}} Mod_{{}^{T}\mathbf{K},x}(\phi)$.~~

The σ-field $\mathcal{M}_T$ of subsets of ${}^{T}\mathbf{K}$ where the measure should be defined is that generated by these sets $Mod_{{}^{T}\mathbf{K},x}(\psi)$.

A measure on this σ-field $\mathcal{M}_T$ invariant under $\sim_{{}^{T}\mathbf{K}}$ is the product measure μ_T of the measure μ defined on the σ-field M generated by the sets $Mod_{\mathbf{K}}(\phi)$ for ϕ a sentence of L, and invariant under $\sim_{\mathbf{K}}$. We then define $P_{\mathbf{K},x}(\psi) =$

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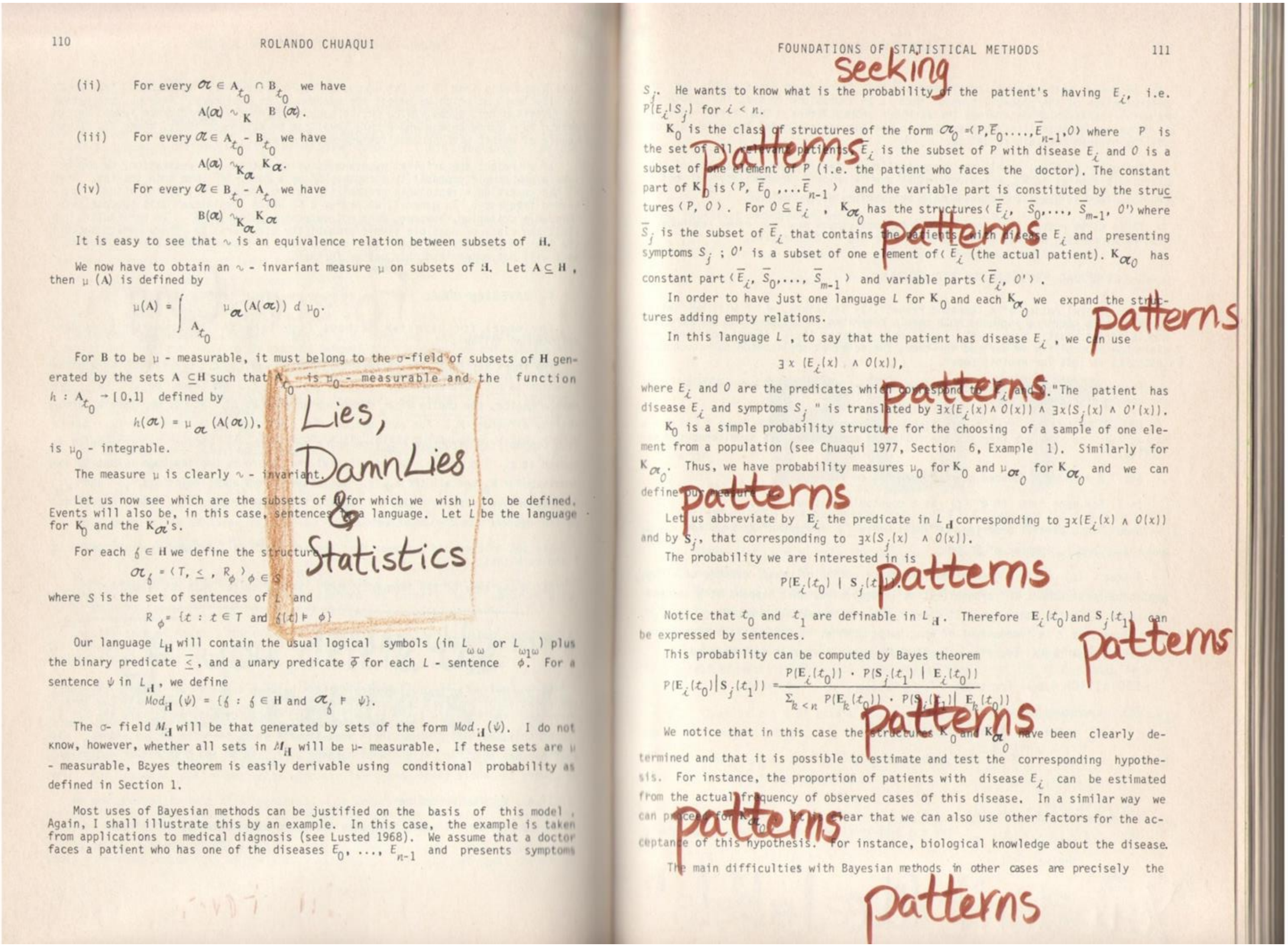

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(ii) For every $\mathfrak{A} \in A_{t_0} \cap B_{t_0}$ we have

$$A(\mathfrak{A}) \sim_K B(\mathfrak{A}).$$

(iii) For every $\mathfrak{A} \in A_{t_0} - B_{t_0}$ we have

$$A(\mathfrak{A}) \sim_{K_{\mathfrak{A}}} K_{\mathfrak{A}}.$$

(iv) For every $\mathfrak{A} \in B_{t_0} - A_{t_0}$ we have

$$B(\mathfrak{A}) \sim_{K_{\mathfrak{A}}} K_{\mathfrak{A}}$$

It is easy to see that $\sim$ is an equivalence relation between subsets of $\mathbb{H}$.

We now have to obtain an $\sim$ - invariant measure μ on subsets of $\mathbb{H}$. Let $A \subseteq \mathbb{H}$, then $\mu(A)$ is defined by

$$\mu(A) = \int_{A_{t_0}} \mu_{\mathfrak{A}}(A(\mathfrak{A}))\, d\,\mu_0.$$

For B to be μ - measurable, it must belong to the σ-field of subsets of $\mathbb{H}$ generated by the sets $A \subseteq \mathbb{H}$ such that A_{t_0} is μ_0 - measurable and the function $h : A_{t_0} \to [0,1]$ defined by

$$h(\mathfrak{A}) = \mu_{\mathfrak{A}}(A(\mathfrak{A})),$$

is μ_0 - integrable.

The measure μ is clearly $\sim$ - invariant.

Let us now see which are the subsets of $\mathbb{H}$ for which we wish μ to be defined. Events will also be, in this case, sentences [illegible] a language. Let L be the language for K_0 and the $K_{\mathfrak{A}}$'s.

For each $\mathfrak{f} \in \mathbb{H}$ we define the structure

$$\mathfrak{A}_{\mathfrak{f}} = \langle T, \leq, R_\phi \rangle_{\phi \in S}$$

where S is the set of sentences of L and

$$R_\phi = \{t : t \in T \text{ and } \mathfrak{f}(t) \models \phi\}$$

Our language $L_{\mathbb{H}}$ will contain the usual logical symbols (in $L_{\omega\omega}$ or $L_{\omega_1\omega}$) plus the binary predicate $\bar{\leq}$, and a unary predicate $\bar{\phi}$ for each L - sentence ϕ. For a sentence ψ in $L_{\mathbb{H}}$, we define

$$Mod_{\mathbb{H}}(\psi) = \{\mathfrak{f} : \mathfrak{f} \in \mathbb{H} \text{ and } \mathfrak{A}_{\mathfrak{f}} \models \psi\}.$$

The σ- field $M_{\mathbb{H}}$ will be that generated by sets of the form $Mod_{\mathbb{H}}(\psi)$. I do not know, however, whether all sets in $M_{\mathbb{H}}$ will be μ- measurable. If these sets are μ - measurable, Bayes theorem is easily derivable using conditional probability as defined in Section 1.

Most uses of Bayesian methods can be justified on the basis of this model. Again, I shall illustrate this by an example. In this case, the example is taken from applications to medical diagnosis (see Lusted 1968). We assume that a doctor faces a patient who has one of the diseases $E_0, \ldots, E_{n-1}$ and presents symptoms

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S_j. He wants to know what is the probability of the patient's having E_i, i.e. $P(E_i | S_j)$ for $i < n$.

K_0 is the class of structures of the form $\mathfrak{A}_0 = \langle P, \bar{E}_0, \ldots, \bar{E}_{n-1}, O \rangle$ where P is the set of all relevant patients, $\bar{E}_i$ is the subset of P with disease E_i and O is a subset of one element of P (i.e. the patient who faces the doctor). The constant part of K_0 is $\langle P, \bar{E}_0, \ldots \bar{E}_{n-1} \rangle$ and the variable part is constituted by the structures $\langle P, O \rangle$. For $O \subseteq E_i$, $K_{\mathfrak{A}_0}$ has the structures $\langle \bar{E}_i, \bar{S}_0, \ldots, \bar{S}_{m-1}, O' \rangle$ where $\bar{S}_j$ is the subset of $\bar{E}_i$ that contains the patients with disease E_i and presenting symptoms S_j; O' is a subset of one element of $\bar{E}_i$ (the actual patient). $K_{\mathfrak{A}_0}$ has constant part $\langle \bar{E}_i, \bar{S}_0, \ldots, \bar{S}_{m-1} \rangle$ and variable parts $\langle \bar{E}_i, O' \rangle$.

In order to have just one language L for K_0 and each $K_{\mathfrak{A}_0}$ we expand the structures adding empty relations.

In this language L, to say that the patient has disease E_i, we can use

$$\exists x\, (E_i(x) \wedge O(x)),$$

where E_i and O are the predicates which correspond to $\bar{E}_i$ and O. "The patient has disease E_i and symptoms S_j" is translated by $\exists x(E_i(x) \wedge O(x)) \wedge \exists x(S_j(x) \wedge O'(x))$.

K_0 is a simple probability structure for the choosing of a sample of one element from a population (see Chuaqui 1977, Section 6, Example 1). Similarly for $K_{\mathfrak{A}_0}$. Thus, we have probability measures μ_0 for K_0 and $\mu_{\mathfrak{A}_0}$ for $K_{\mathfrak{A}_0}$ and we can define our measure [illegible]

Let us abbreviate by $\mathbf{E}_i$ the predicate in $L_{\mathbb{H}}$ corresponding to $\exists x(E_i(x) \wedge O(x))$ and by $\mathbf{S}_j$, that corresponding to $\exists x(S_j(x) \wedge O(x))$.

The probability we are interested in is

$$P(\mathbf{E}_i(t_0) \mid \mathbf{S}_j(t_1)).$$

Notice that t_0 and t_1 are definable in $L_{\mathbb{H}}$. Therefore $\mathbf{E}_i(t_0)$ and $\mathbf{S}_j(t_1)$ can be expressed by sentences.

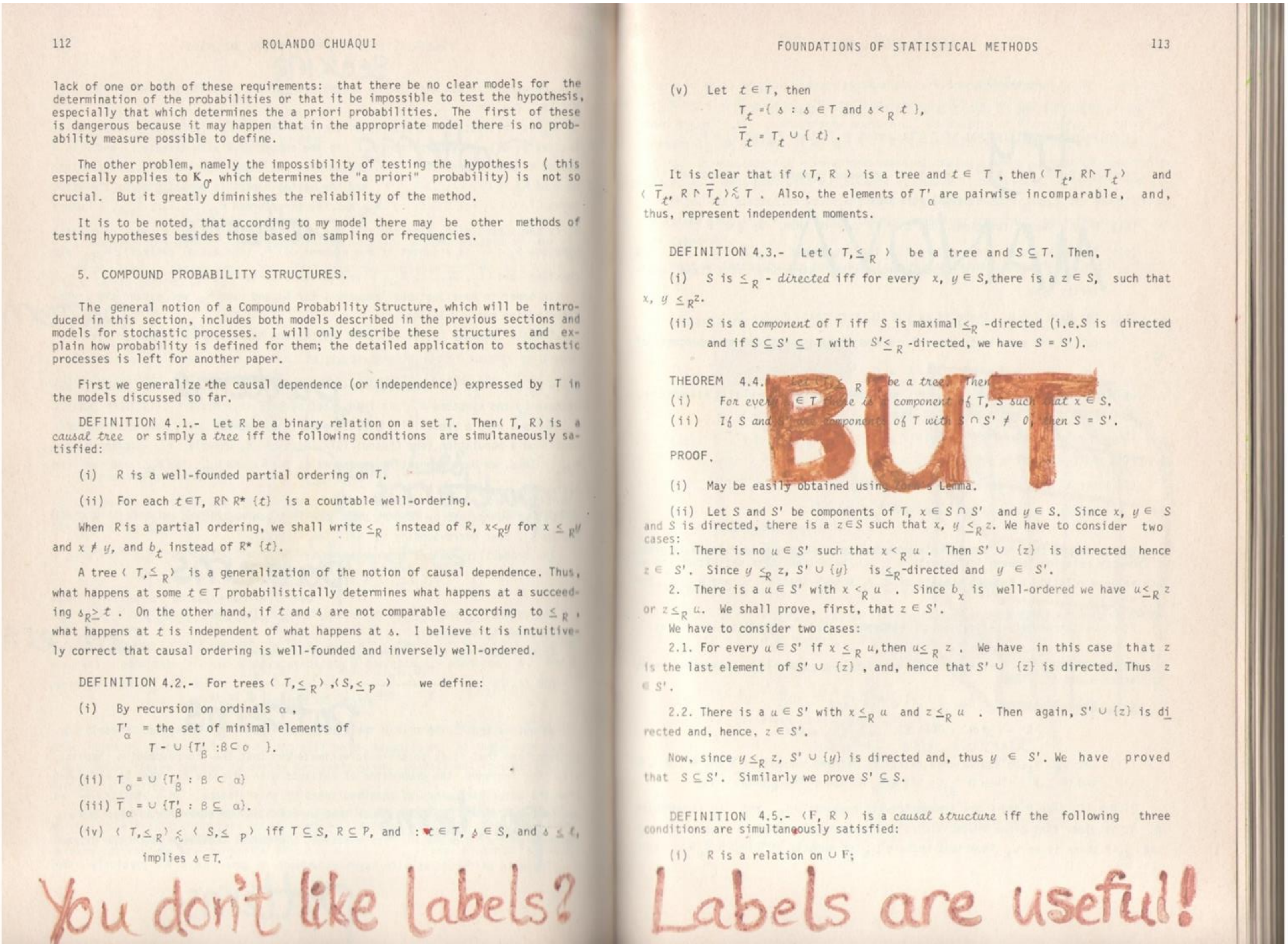
This probability can be computed by Bayes theorem

$$P(\mathbf{E}_i(t_0) | \mathbf{S}_j(t_1)) = \frac{P(\mathbf{E}_i(t_0)) \cdot P(\mathbf{S}_j(t_1) \mid \mathbf{E}_i(t_0))}{\Sigma_{k<n} P(\mathbf{E}_k(t_0)) \cdot P(\mathbf{S}_j(t_1) | \mathbf{E}_k(t_0))}$$

We notice that in this case the structures K_0 and $K_{\mathfrak{A}_0}$ have been clearly determined and that it is possible to estimate and test the corresponding hypothesis. For instance, the proportion of patients with disease E_i can be estimated from the actual frequency of observed cases of this disease. In a similar way we can proceed for $K_{\mathfrak{A}_0}$. It is clear that we can also use other factors for the acceptance of this hypothesis. For instance, biological knowledge about the disease.

The main difficulties with Bayesian methods in other cases are precisely the

Mathematical Logic in Catholic North America, altered book, 2013, pp. 110-111.



lack of one or both of these requirements: that there be no clear models for the determination of the probabilities or that it be impossible to test the hypothesis, especially that which determines the a priori probabilities. The first of these is dangerous because it may happen that in the appropriate model there is no probability measure possible to define.

The other problem, namely the impossibility of testing the hypothesis (this especially applies to K_σ which determines the "a priori" probability) is not so crucial. But it greatly diminishes the reliability of the method.

It is to be noted, that according to my model there may be other methods of testing hypotheses besides those based on sampling or frequencies.

5. COMPOUND PROBABILITY STRUCTURES.

The general notion of a Compound Probability Structure, which will be introduced in this section, includes both models described in the previous sections and models for stochastic processes. I will only describe these structures and explain how probability is defined for them; the detailed application to stochastic processes is left for another paper.

First we generalize the causal dependence (or independence) expressed by T in the models discussed so far.

DEFINITION 4.1.- Let R be a binary relation on a set T. Then $\langle T, R\rangle$ is a *causal tree* or simply a *tree* iff the following conditions are simultaneously satisfied:

(i) R is a well-founded partial ordering on T.

(ii) For each $t \in T$, $R \upharpoonright R^* \{t\}$ is a countable well-ordering.

When R is a partial ordering, we shall write $\leq_R$ instead of R, $x <_R y$ for $x \leq_R y$ and $x \neq y$, and b_t instead of $R^* \{t\}$.

A tree $\langle T, \leq_R\rangle$ is a generalization of the notion of causal dependence. Thus, what happens at some $t \in T$ probabilistically determines what happens at a succeeding $s \geq_R t$. On the other hand, if t and s are not comparable according to $\leq_R$, what happens at t is independent of what happens at s. I believe it is intuitively correct that causal ordering is well-founded and inversely well-ordered.

DEFINITION 4.2.- For trees $\langle T, \leq_R\rangle$, $\langle S, \leq_P\rangle$ we define:

(i) By recursion on ordinals α,
T'_α = the set of minimal elements of
$T - \cup \{T'_\beta : \beta \subset \alpha\}$.

(ii) $T_\alpha = \cup \{T'_\beta : \beta \subset \alpha\}$

(iii) $\overline{T}_\alpha = \cup \{T'_\beta : \beta \subseteq \alpha\}$.

(iv) $\langle T, \leq_R\rangle \lesssim \langle S, \leq_P\rangle$ iff $T \subseteq S$, $R \subseteq P$, and $t \in T$, $s \in S$, and $s \leq t$, implies $s \in T$.

You don't like labels?

(v) Let $t \in T$, then
$T_t = \{ s : s \in T \text{ and } s <_R t \}$,
$\overline{T}_t = T_t \cup \{ t \}$.

It is clear that if $\langle T, R\rangle$ is a tree and $t \in T$, then $\langle T_t, R \upharpoonright T_t\rangle$ and $\langle \overline{T}_t, R \upharpoonright \overline{T}_t\rangle \lesssim T$. Also, the elements of T'_α are pairwise incomparable, and, thus, represent independent moments.

DEFINITION 4.3.- Let $\langle T, \leq_R\rangle$ be a tree and $S \subseteq T$. Then,

(i) S is $\leq_R$ - *directed* iff for every $x, y \in S$, there is a $z \in S$, such that $x, y \leq_R z$.

(ii) S is a *component* of T iff S is maximal $\leq_R$ -directed (i.e. S is directed and if $S \subseteq S' \subseteq T$ with S' $\leq_R$ -directed, we have $S = S'$).

BUT

THEOREM 4.4. [illegible] $_R$ [illegible] *be a tree. Then*
(i) *For eve*[illegible] $\in T$ [illegible]*e* [illegible] *component* [illegible] *T*, [illegible] *at* $x \in S$.
(ii) *If S and* [illegible] *components of T with* [illegible] $\cap S' \neq 0$, [illegible]*en* $S = S'$.

PROOF.

(i) May be easily obtained usin[illegible] Lemma.

(ii) Let S and S' be components of T, $x \in S \cap S'$ and $y \in S$. Since $x, y \in S$ and S is directed, there is a $z \in S$ such that $x, y \leq_R z$. We have to consider two cases:

1. There is no $u \in S'$ such that $x <_R u$. Then $S' \cup \{z\}$ is directed hence $z \in S'$. Since $y \leq_R z$, $S' \cup \{y\}$ is $\leq_R$-directed and $y \in S'$.

2. There is a $u \in S'$ with $x <_R u$. Since b_x is well-ordered we have $u \leq_R z$ or $z \leq_R u$. We shall prove, first, that $z \in S'$.
We have to consider two cases:

2.1. For every $u \in S'$ if $x \leq_R u$, then $u \leq_R z$. We have in this case that z is the last element of $S' \cup \{z\}$, and, hence that $S' \cup \{z\}$ is directed. Thus $z \in S'$.

2.2. There is a $u \in S'$ with $x \leq_R u$ and $z \leq_R u$. Then again, $S' \cup \{z\}$ is directed and, hence, $z \in S'$.

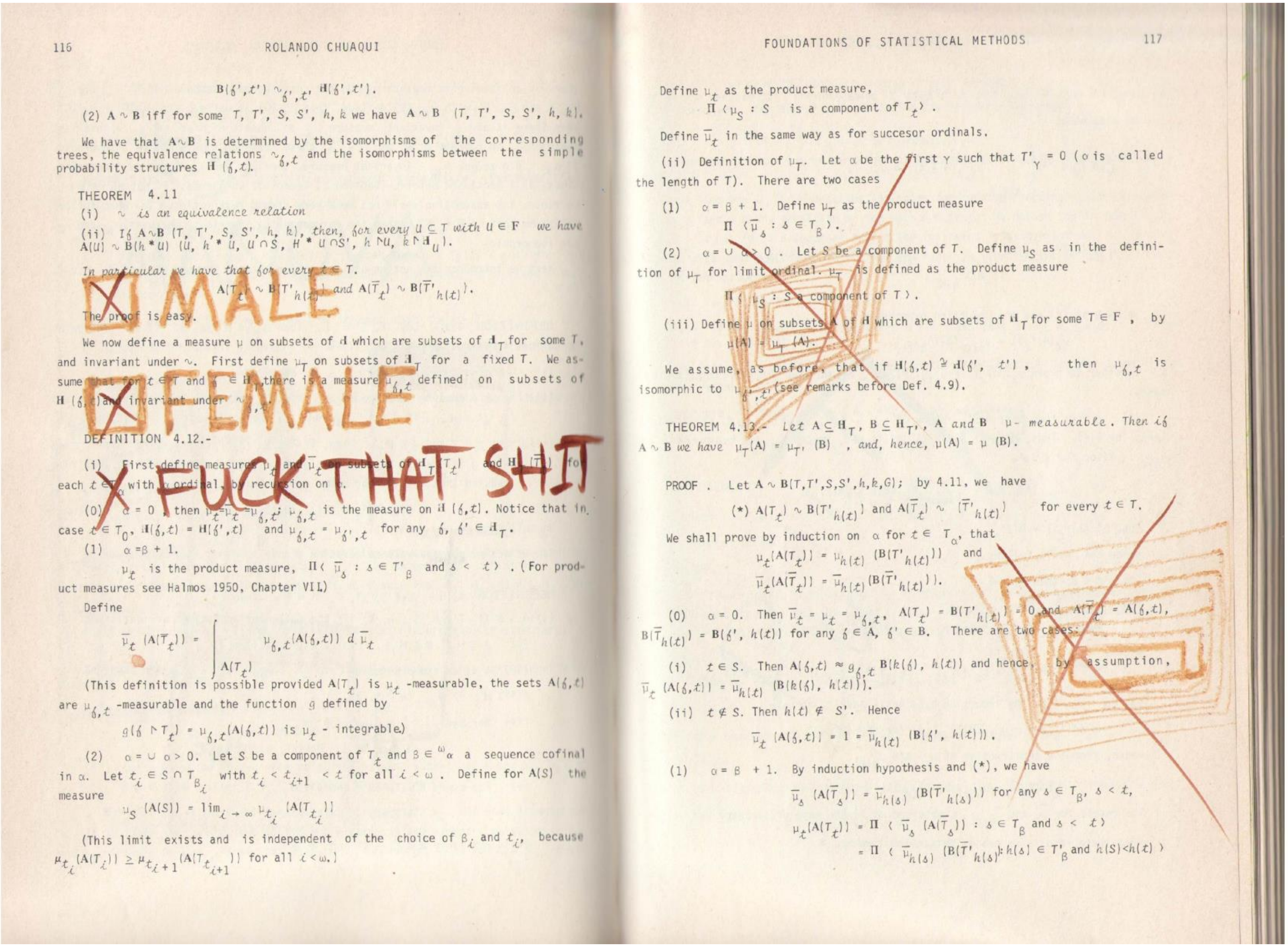
Now, since $y \leq_R z$, $S' \cup \{y\}$ is directed and, thus $y \in S'$. We have proved that $S \subseteq S'$. Similarly we prove $S' \subseteq S$.

DEFINITION 4.5.- $\langle F, R\rangle$ is a *causal structure* iff the following three conditions are simultaneously satisfied:

(i) R is a relation on $\cup F$;

Labels are useful!

Mathematical Logic in Catholic North America, altered book, 2013, pp. 112-113.



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$B(s',t') \sim_{s',t'} H(s',t')$.

(2) $A \sim B$ iff for some T, T', S, S', h, k we have $A \sim B$ (T, T', S, S', h, k).

We have that $A \sim B$ is determined by the isomorphisms of the corresponding trees, the equivalence relations $\sim_{s,t}$ and the isomorphisms between the simple probability structures $H(s,t)$.

THEOREM 4.11

(i) $\sim$ *is an equivalence relation*

(ii) *If* $A \sim B$ (T, T', S, S', h, k), *then, for every* $U \subseteq T$ *with* $U \in F$ *we have* $A(U) \sim B(h^*U)$ $(U, h^*U, U \cap S, H^*U \cap S', h \restriction U, k \restriction H_U)$.

In particular we have that for every $t \in T$.

$A(T_t) \sim B(T'_{h(t)})$ *and* $A(\bar{T}_t) \sim B(\bar{T}'_{h(t)})$.

MALE

The proof is easy.

We now define a measure μ on subsets of H which are subsets of H_T for some T, and invariant under $\sim$. First define μ_T on subsets of H_T for a fixed T. We assume that for $t \in T$ and $s \in H_t$ there is a measure $\mu_{s,t}$ defined on subsets of $H(s,t)$ and invariant under $\sim$.

FEMALE

DEFINITION 4.12.-

FUCK THAT SHIT

(i) First define measures μ_t and $\bar{\mu}_t$ on subsets of $H_T(T_t)$ and $H_T(\bar{T}_t)$ for each $t \in T_\alpha$ with α ordinal, by recursion on α.

(0) $\alpha = 0$, then $\mu_t = \bar{\mu}_t = \mu_{s,t}$, $\mu_{s,t}$ is the measure on $H(s,t)$. Notice that in case $t \in T_0$, $H(s,t) = H(s',t)$ and $\mu_{s,t} = \mu_{s',t}$ for any $s, s' \in H_T$.

(1) $\alpha = \beta + 1$.

μ_t is the product measure, $\Pi\langle \bar{\mu}_s : s \in T'_\beta$ and $s < t\rangle$. (For product measures see Halmos 1950, Chapter VII.)

Define

$$\bar{\mu}_t(A(\bar{T}_t)) = \int_{A(T_t)} \mu_{s,t}(A(s,t))\, d\bar{\mu}_t$$

(This definition is possible provided $A(T_t)$ is μ_t-measurable, the sets $A(s,t)$ are $\mu_{s,t}$-measurable and the function g defined by

$g(s \restriction T_t) = \mu_{s,t}(A(s,t))$ is μ_t-integrable.)

(2) $\alpha = \cup\alpha > 0$. Let S be a component of T_t and $\beta \in {}^{\omega}\alpha$ a sequence cofinal in α. Let $t_i \in S \cap T_{\beta_i}$ with $t_i < t_{i+1} < t$ for all $i < \omega$. Define for $A(S)$ the measure

$$\mu_S(A(S)) = \lim_{i \to \infty} \mu_{t_i}(A(T_{t_i}))$$

(This limit exists and is independent of the choice of β_i and t_i, because $\mu_{t_i}(A(T_{t_i})) \geq \mu_{t_{i+1}}(A(T_{t_{i+1}}))$ for all $i < \omega$.)

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Define μ_t as the product measure,

$\Pi\langle \mu_S : S$ is a component of $T_t\rangle$.

Define $\bar{\mu}_t$ in the same way as for succesor ordinals.

(ii) Definition of μ_T. Let α be the first γ such that $T'_\gamma = 0$ (α is called the length of T). There are two cases

(1) $\alpha = \beta + 1$. Define μ_T as the product measure $\Pi\langle \bar{\mu}_s : s \in T_\beta\rangle$.

(2) $\alpha = \cup\alpha > 0$. Let S be a component of T. Define μ_S as in the definition of μ_t for limit ordinal. μ_T is defined as the product measure $\Pi\langle \mu_S : S$ a component of $T\rangle$.

(iii) Define μ on subsets A of H which are subsets of H_T for some $T \in F$, by $\mu(A) = \mu_T(A)$.

We assume, as before, that if $H(s,t) \cong H(s',t')$, then $\mu_{s,t}$ is isomorphic to $\mu_{s',t'}$ (see remarks before Def. 4.9).

THEOREM 4.13.- *Let* $A \subseteq H_T$, $B \subseteq H_{T'}$, A *and* B μ-*measurable. Then if* $A \sim B$ *we have* $\mu_T(A) = \mu_{T'}(B)$, *and, hence,* $\mu(A) = \mu(B)$.

PROOF. Let $A \sim B(T,T',S,S',h,k,G)$; by 4.11, we have

(*) $A(T_t) \sim B(T'_{h(t)})$ and $A(\bar{T}_t) \sim (\bar{T}'_{h(t)})$ for every $t \in T$.

We shall prove by induction on α for $t \in T_\alpha$, that

$\mu_t(A(T_t)) = \mu_{h(t)}(B(T'_{h(t)}))$ and

$\bar{\mu}_t(A(\bar{T}_t)) = \bar{\mu}_{h(t)}(B(\bar{T}'_{h(t)}))$.

(0) $\alpha = 0$. Then $\bar{\mu}_t = \mu_t = \mu_{s,t}$, $A(T_t) = B(T'_{h(t)}) = 0$ and $A(\bar{T}_t) = A(s,t)$, $B(\bar{T}_{h(t)}) = B(s', h(t))$ for any $s \in A$, $s' \in B$. There are two cases:

(i) $t \in S$. Then $A(s,t) \approx g_{s,t} B(k(s), h(t))$ and hence, by assumption, $\bar{\mu}_t(A(s,t)) = \bar{\mu}_{h(t)}(B(k(s), h(t)))$.

(ii) $t \notin S$. Then $h(t) \notin S'$. Hence

$\bar{\mu}_t(A(s,t)) = 1 = \bar{\mu}_{h(t)}(B(s', h(t)))$.

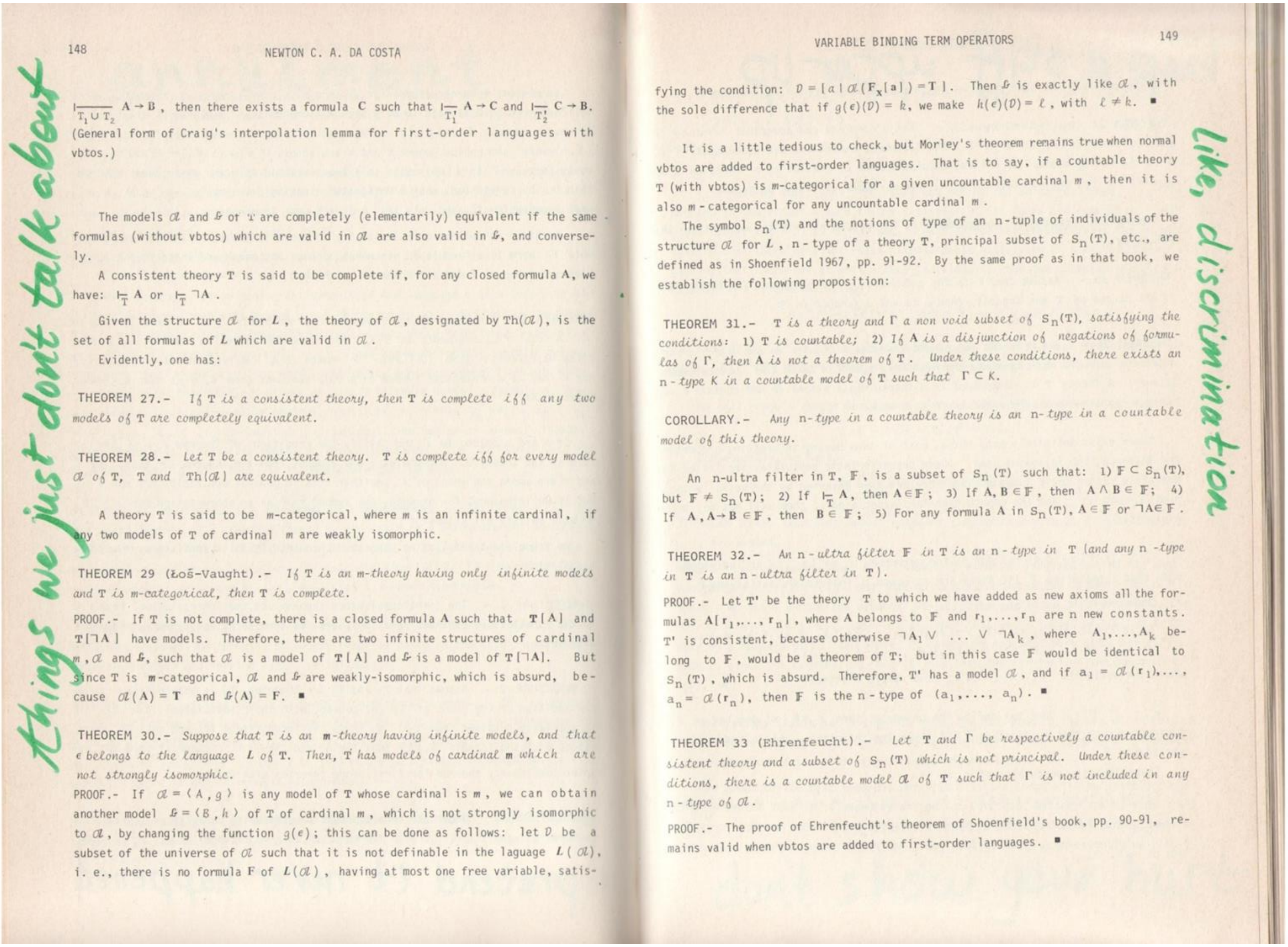
(1) $\alpha = \beta + 1$. By induction hypothesis and (*), we have

$\bar{\mu}_s(A(\bar{T}_s)) = \bar{\mu}_{h(s)}(B(\bar{T}'_{h(s)}))$ for any $s \in T_\beta$, $s < t$,

$\mu_t(A(T_t)) = \Pi\langle \bar{\mu}_s(A(\bar{T}_s)) : s \in T_\beta$ and $s < t\rangle$

$= \Pi\langle \bar{\mu}_{h(s)}(B(\bar{T}'_{h(s)})) : h(s) \in T'_\beta$ and $h(S) < h(t)\rangle$

Mathematical Logic in Catholic North America, altered book, 2013, pp. 116-117.



things we just don't talk about

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$\vdash_{T_1 \cup T_2} A \to B$, then there exists a formula C such that $\vdash_{T_1'} A \to C$ and $\vdash_{T_2'} C \to B$. (General form of Craig's interpolation lemma for first-order languages with vbtos.)

The models $\mathfrak{A}$ and $\mathfrak{B}$ of T are completely (elementarily) equivalent if the same formulas (without vbtos) which are valid in $\mathfrak{A}$ are also valid in $\mathfrak{B}$, and conversely.

A consistent theory T is said to be complete if, for any closed formula A, we have: $\vdash_T A$ or $\vdash_T \neg A$.

Given the structure $\mathfrak{A}$ for L, the theory of $\mathfrak{A}$, designated by $Th(\mathfrak{A})$, is the set of all formulas of L which are valid in $\mathfrak{A}$.

Evidently, one has:

THEOREM 27.- *If T is a consistent theory, then T is complete iff any two models of T are completely equivalent.*

THEOREM 28.- *Let T be a consistent theory. T is complete iff for every model $\mathfrak{A}$ of T, T and $Th(\mathfrak{A})$ are equivalent.*

A theory T is said to be m-categorical, where m is an infinite cardinal, if any two models of T of cardinal m are weakly isomorphic.

THEOREM 29 (Łoś-Vaught).- *If T is an m-theory having only infinite models and T is m-categorical, then T is complete.*

PROOF.- If T is not complete, there is a closed formula A such that $T[A]$ and $T[\neg A]$ have models. Therefore, there are two infinite structures of cardinal m, $\mathfrak{A}$ and $\mathfrak{B}$, such that $\mathfrak{A}$ is a model of $T[A]$ and $\mathfrak{B}$ is a model of $T[\neg A]$. But since T is m-categorical, $\mathfrak{A}$ and $\mathfrak{B}$ are weakly-isomorphic, which is absurd, because $\mathfrak{A}(A) = T$ and $\mathfrak{B}(A) = F$. ∎

THEOREM 30.- *Suppose that T is an m-theory having infinite models, and that ϵ belongs to the language L of T. Then, T has models of cardinal m which are not strongly isomorphic.*

PROOF.- If $\mathfrak{A} = \langle A, g \rangle$ is any model of T whose cardinal is m, we can obtain another model $\mathfrak{B} = \langle B, h \rangle$ of T of cardinal m, which is not strongly isomorphic to $\mathfrak{A}$, by changing the function $g(\epsilon)$; this can be done as follows: let D be a subset of the universe of $\mathfrak{A}$ such that it is not definable in the laguage $L(\mathfrak{A})$, i. e., there is no formula F of $L(\mathfrak{A})$, having at most one free variable, satis-

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fying the condition: $D = [a \mid \mathfrak{A}(F_x[a]) = T]$. Then $\mathfrak{B}$ is exactly like $\mathfrak{A}$, with the sole difference that if $g(\epsilon)(D) = k$, we make $h(\epsilon)(D) = \ell$, with $\ell \neq k$. ∎

It is a little tedious to check, but Morley's theorem remains true when normal vbtos are added to first-order languages. That is to say, if a countable theory T (with vbtos) is m-categorical for a given uncountable cardinal m, then it is also m-categorical for any uncountable cardinal m.

The symbol $S_n(T)$ and the notions of type of an n-tuple of individuals of the structure $\mathfrak{A}$ for L, n-type of a theory T, principal subset of $S_n(T)$, etc., are defined as in Shoenfield 1967, pp. 91-92. By the same proof as in that book, we establish the following proposition:

THEOREM 31.- *T is a theory and Γ a non void subset of $S_n(T)$, satisfying the conditions: 1) T is countable; 2) If A is a disjunction of negations of formulas of Γ, then A is not a theorem of T. Under these conditions, there exists an n-type K in a countable model of T such that Γ ⊂ K.*

COROLLARY.- *Any n-type in a countable theory is an n-type in a countable model of this theory.*

An n-ultra filter in T, $\mathbb{F}$, is a subset of $S_n(T)$ such that: 1) $\mathbb{F} \subset S_n(T)$, but $\mathbb{F} \neq S_n(T)$; 2) If $\vdash_T A$, then $A \in \mathbb{F}$; 3) If $A, B \in \mathbb{F}$, then $A \wedge B \in \mathbb{F}$; 4) If $A, A \to B \in \mathbb{F}$, then $B \in \mathbb{F}$; 5) For any formula A in $S_n(T)$, $A \in \mathbb{F}$ or $\neg A \in \mathbb{F}$.

THEOREM 32.- *An n-ultra filter $\mathbb{F}$ in T is an n-type in T (and any n-type in T is an n-ultra filter in T).*

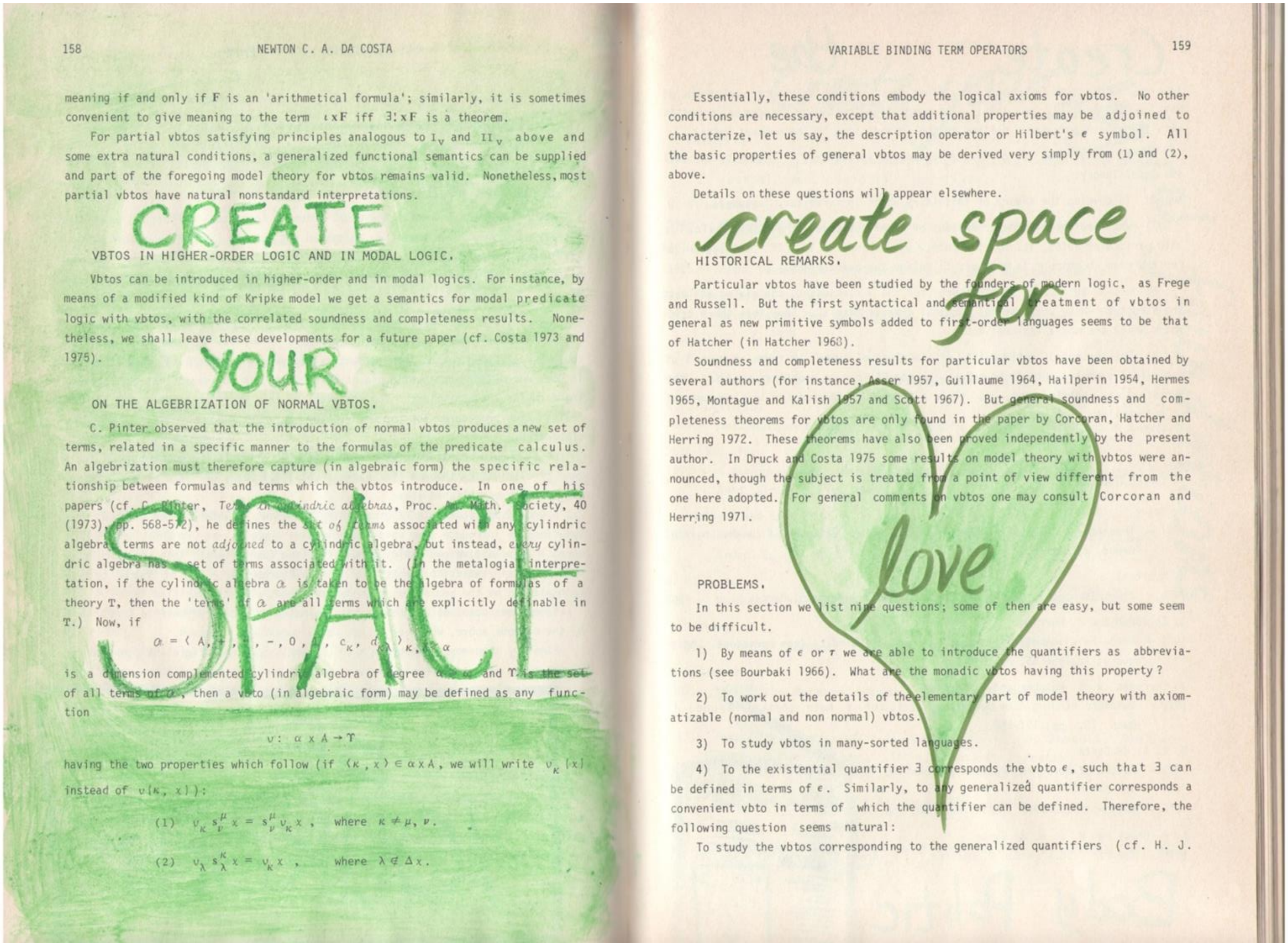
PROOF.- Let T' be the theory T to which we have added as new axioms all the formulas $A[r_1, \ldots, r_n]$, where A belongs to $\mathbb{F}$ and $r_1, \ldots, r_n$ are n new constants. T' is consistent, because otherwise $\neg A_1 \vee \ldots \vee \neg A_k$, where $A_1, \ldots, A_k$ belong to $\mathbb{F}$, would be a theorem of T; but in this case $\mathbb{F}$ would be identical to $S_n(T)$, which is absurd. Therefore, T' has a model $\mathfrak{A}$, and if $a_1 = \mathfrak{A}(r_1), \ldots, a_n = \mathfrak{A}(r_n)$, then $\mathbb{F}$ is the n-type of $(a_1, \ldots, a_n)$. ∎

THEOREM 33 (Ehrenfeucht).- *Let T and Γ be respectively a countable consistent theory and a subset of $S_n(T)$ which is not principal. Under these conditions, there is a countable model $\mathfrak{A}$ of T such that Γ is not included in any n-type of $\mathfrak{A}$.*

PROOF.- The proof of Ehrenfeucht's theorem of Shoenfield's book, pp. 90-91, remains valid when vbtos are added to first-order languages. ∎

like, discrimination

Mathematical Logic in Catholic North America, altered book, 2013, pp. 148-149.



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meaning if and only if F is an 'arithmetical formula'; similarly, it is sometimes convenient to give meaning to the term $\iota x F$ iff $\exists ! x F$ is a theorem.

For partial vbtos satisfying principles analogous to I_v and II_v above and some extra natural conditions, a generalized functional semantics can be supplied and part of the foregoing model theory for vbtos remains valid. Nonetheless, most partial vbtos have natural nonstandard interpretations.

VBTOS IN HIGHER-ORDER LOGIC AND IN MODAL LOGIC.

Vbtos can be introduced in higher-order and in modal logics. For instance, by means of a modified kind of Kripke model we get a semantics for modal predicate logic with vbtos, with the correlated soundness and completeness results. Nonetheless, we shall leave these developments for a future paper (cf. Costa 1973 and 1975).

ON THE ALGEBRIZATION OF NORMAL VBTOS.

C. Pinter observed that the introduction of normal vbtos produces a new set of terms, related in a specific manner to the formulas of the predicate calculus. An algebrization must therefore capture (in algebraic form) the specific relationship between formulas and terms which the vbtos introduce. In one of his papers (cf. C. Pinter, *Terms in cylindric algebras*, Proc. Am. Math. Society, 40 (1973), pp. 568-572), he defines the *set of terms* associated with any cylindric algebra: terms are not *adjoined* to a cylindric algebra, but instead, *every* cylindric algebra has a set of terms associated with it. (In the metalogical interpretation, if the cylindric algebra $\mathfrak{a}$ is taken to be the algebra of formulas of a theory T, then the 'terms' of $\mathfrak{a}$ are all terms which are explicitly definable in T.) Now, if

$$\mathfrak{a} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$$

is a dimension complemented cylindric algebra of degree $\alpha \geq \omega$ and T is the set of all terms of $\mathfrak{a}$, then a vbto (in algebraic form) may be defined as any function

$$v : \alpha \times A \to T$$

having the two properties which follow (if $\langle \kappa, x \rangle \in \alpha \times A$, we will write $v_\kappa[x]$ instead of $v[\kappa, x]$):

$$(1) \quad v_\kappa s^\mu_\nu x = s^\mu_\nu v_\kappa x, \quad \text{where } \kappa \neq \mu, \nu.$$

$$(2) \quad v_\lambda s^\kappa_\lambda x = v_\kappa x, \quad \text{where } \lambda \notin \Delta x.$$

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Essentially, these conditions embody the logical axioms for vbtos. No other conditions are necessary, except that additional properties may be adjoined to characterize, let us say, the description operator or Hilbert's ϵ symbol. All the basic properties of general vbtos may be derived very simply from (1) and (2), above.

Details on these questions will appear elsewhere.

HISTORICAL REMARKS.

Particular vbtos have been studied by the founders of modern logic, as Frege and Russell. But the first syntactical and semantical treatment of vbtos in general as new primitive symbols added to first-order languages seems to be that of Hatcher (in Hatcher 1968).

Soundness and completeness results for particular vbtos have been obtained by several authors (for instance, Asser 1957, Guillaume 1964, Hailperin 1954, Hermes 1965, Montague and Kalish 1957 and Scott 1967). But general soundness and completeness theorems for vbtos are only found in the paper by Corcoran, Hatcher and Herring 1972. These theorems have also been proved independently by the present author. In Druck and Costa 1975 some results on model theory with vbtos were announced, though the subject is treated from a point of view different from the one here adopted. For general comments on vbtos one may consult Corcoran and Herring 1971.

PROBLEMS.

In this section we list nine questions; some of then are easy, but some seem to be difficult.

1) By means of ϵ or τ we are able to introduce the quantifiers as abbreviations (see Bourbaki 1966). What are the monadic vbtos having this property?

2) To work out the details of the elementary part of model theory with axiomatizable (normal and non normal) vbtos.

3) To study vbtos in many-sorted languages.

4) To the existential quantifier $\exists$ corresponds the vbto ϵ, such that $\exists$ can be defined in terms of ϵ. Similarly, to any generalized quantifier corresponds a convenient vbto in terms of which the quantifier can be defined. Therefore, the following question seems natural:

To study the vbtos corresponding to the generalized quantifiers (cf. H. J.

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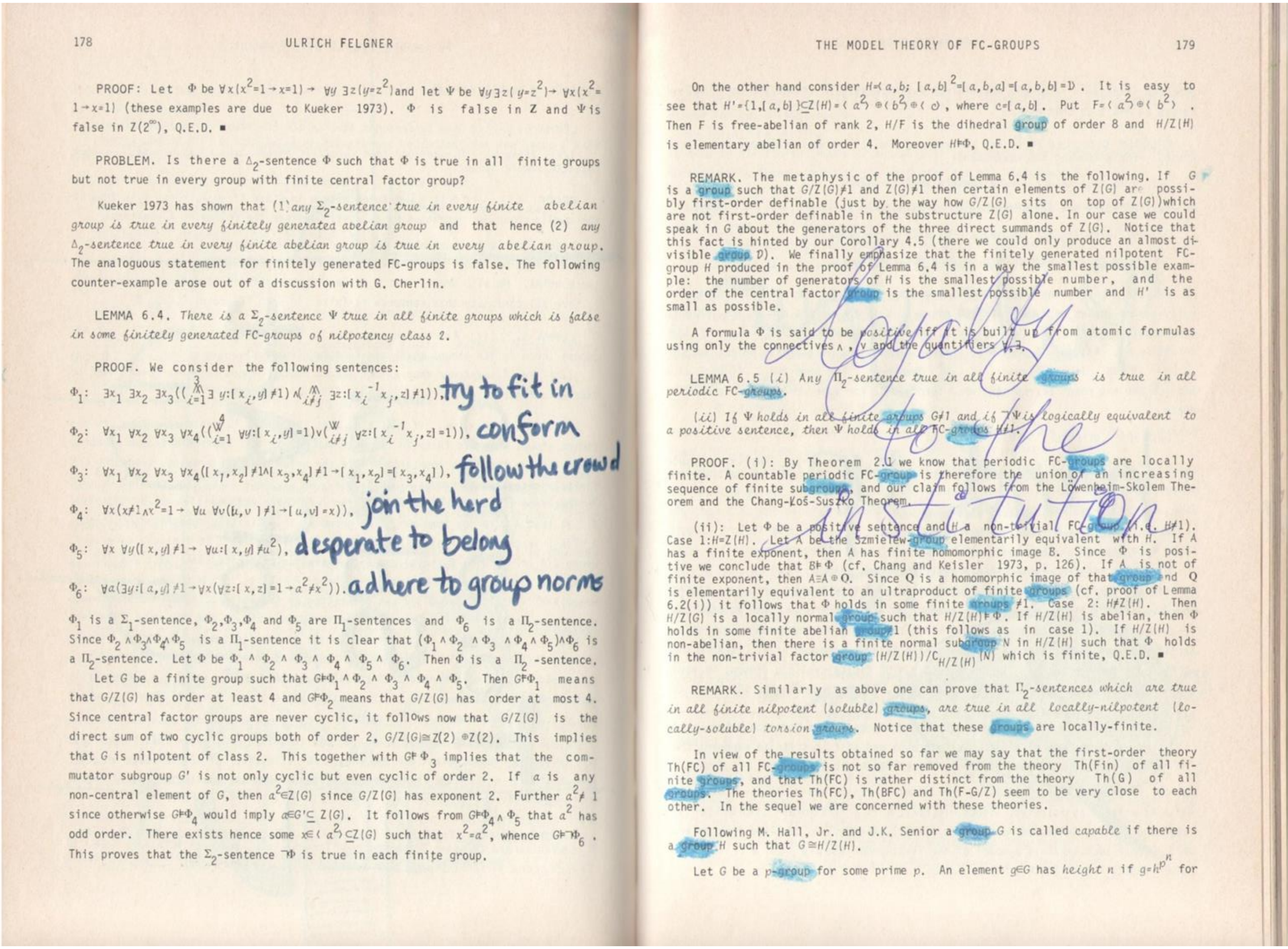

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PROOF: Let Φ be $\forall x(x^2=1 \to x=1) \to \forall y\, \exists z(y=z^2)$ and let Ψ be $\forall y \exists z(y=z^2) \to \forall x(x^2=1 \to x=1)$ (these examples are due to Kueker 1973). Φ is false in Z and Ψ is false in $Z(2^\infty)$, Q.E.D. ■

PROBLEM. Is there a Δ_2-sentence Φ such that Φ is true in all finite groups but not true in every group with finite central factor group?

Kueker 1973 has shown that (1) *any Σ_2-sentence true in every finite abelian group is true in every finitely generated abelian group* and that hence (2) *any Δ_2-sentence true in every finite abelian group is true in every abelian group*. The analoguous statement for finitely generated FC-groups is false. The following counter-example arose out of a discussion with G. Cherlin.

LEMMA 6.4. *There is a Σ_2-sentence Ψ true in all finite groups which is false in some finitely generated FC-groups of nilpotency class 2.*

PROOF. We consider the following sentences:

Φ_1: $\exists x_1 \exists x_2 \exists x_3((\bigwedge_{i=1}^{3} \exists y:[x_i,y]\neq 1) \wedge (\bigwedge_{i\neq j} \exists z:[x_i^{-1}x_j,z]\neq 1))$, try to fit in

Φ_2: $\forall x_1 \forall x_2 \forall x_3 \forall x_4((\bigvee_{i=1}^{4} \forall y:[x_i,y]=1) \vee (\bigvee_{i\neq j} \forall z:[x_i^{-1}x_j,z]=1))$, conform

Φ_3: $\forall x_1 \forall x_2 \forall x_3 \forall x_4([x_1,x_2]\neq 1 \wedge [x_3,x_4]\neq 1 \to [x_1,x_2]=[x_3,x_4])$, follow the crowd

Φ_4: $\forall x(x\neq 1 \wedge x^2=1 \to \forall u\, \forall v([u,v]\neq 1 \to [u,v]=x))$, join the herd

Φ_5: $\forall x\, \forall y([x,y]\neq 1 \to \forall u:[x,y]\neq u^2)$, desperate to belong

Φ_6: $\forall a(\exists y:[a,y]\neq 1 \to \forall x(\forall z:[x,z]=1 \to a^2\neq x^2))$. adhere to group norms

Φ_1 is a Σ_1-sentence, Φ_2,Φ_3,Φ_4 and Φ_5 are Π_1-sentences and Φ_6 is a Π_2-sentence. Since $\Phi_2 \wedge \Phi_3 \wedge \Phi_4 \wedge \Phi_5$ is a Π_1-sentence it is clear that $(\Phi_1 \wedge \Phi_2 \wedge \Phi_3 \wedge \Phi_4 \wedge \Phi_5) \wedge \Phi_6$ is a Π_2-sentence. Let Φ be $\Phi_1 \wedge \Phi_2 \wedge \Phi_3 \wedge \Phi_4 \wedge \Phi_5 \wedge \Phi_6$. Then Φ is a Π_2-sentence.

Let G be a finite group such that $G \models \Phi_1 \wedge \Phi_2 \wedge \Phi_3 \wedge \Phi_4 \wedge \Phi_5$. Then $G \models \Phi_1$ means that $G/Z(G)$ has order at least 4 and $G \models \Phi_2$ means that $G/Z(G)$ has order at most 4. Since central factor groups are never cyclic, it follows now that $G/Z(G)$ is the direct sum of two cyclic groups both of order 2, $G/Z(G) \cong Z(2) \oplus Z(2)$. This implies that G is nilpotent of class 2. This together with $G \models \Phi_3$ implies that the commutator subgroup G' is not only cyclic but even cyclic of order 2. If a is any non-central element of G, then $a^2 \in Z(G)$ since $G/Z(G)$ has exponent 2. Further $a^2 \neq 1$ since otherwise $G \models \Phi_4$ would imply $a \in G' \subseteq Z(G)$. It follows from $G \models \Phi_4 \wedge \Phi_5$ that a^2 has odd order. There exists hence some $x \in \langle a^2 \rangle \subseteq Z(G)$ such that $x^2=a^2$, whence $G \models \neg\Phi_6$. This proves that the Σ_2-sentence $\neg\Phi$ is true in each finite group.

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On the other hand consider $H=\langle a,b;\ [a,b]^2=[a,b,a]=[a,b,b]=1\rangle$. It is easy to see that $H'=\{1,[a,b]\} \subseteq Z(H)=\langle a^2\rangle \oplus \langle b^2\rangle \oplus \langle c\rangle$, where $c=[a,b]$. Put $F=\langle a^2\rangle \oplus \langle b^2\rangle$. Then F is free-abelian of rank 2, H/F is the dihedral group of order 8 and $H/Z(H)$ is elementary abelian of order 4. Moreover $H \models \Phi$, Q.E.D. ■

REMARK. The metaphysic of the proof of Lemma 6.4 is the following. If G is a group such that $G/Z(G)\neq 1$ and $Z(G)\neq 1$ then certain elements of $Z(G)$ are possibly first-order definable (just by the way how $G/Z(G)$ sits on top of $Z(G)$) which are not first-order definable in the substructure $Z(G)$ alone. In our case we could speak in G about the generators of the three direct summands of $Z(G)$. Notice that this fact is hinted by our Corollary 4.5 (there we could only produce an almost divisible group D). We finally emphasize that the finitely generated nilpotent FC-group H produced in the proof of Lemma 6.4 is in a way the smallest possible example: the number of generators of H is the smallest possible number, and the order of the central factor group is the smallest possible number and H' is as small as possible.

Loyalty

A formula Φ is said to be *positive* iff it is built up from atomic formulas using only the connectives $\wedge$, $\vee$ and the quantifiers $\forall, \exists$.

LEMMA 6.5 (*i*) *Any Π_2-sentence true in all finite groups is true in all periodic FC-groups.*

to the

(*ii*) *If Ψ holds in all finite groups $G\neq 1$ and if $\neg\Psi$ is logically equivalent to a positive sentence, then Ψ holds in all FC-groups $H\neq 1$.*

PROOF. (i): By Theorem 2.1 we know that periodic FC-groups are locally finite. A countable periodic FC-group is therefore the union of an increasing sequence of finite subgroups, and our claim follows from the Löwenheim-Skolem Theorem and the Chang-Łoś-Suszko Theorem.

institution

(ii): Let Φ be a positive sentence and H a non-trivial FC-group (i.e. $H\neq 1$). Case 1: $H=Z(H)$. Let A be the Szmielew-group elementarily equivalent with H. If A has a finite exponent, then A has finite homomorphic image B. Since Φ is positive we conclude that $B \models \Phi$ (cf. Chang and Keisler 1973, p. 126). If A is not of finite exponent, then $A \equiv A \oplus \mathbb{Q}$. Since $\mathbb{Q}$ is a homomorphic image of that group and $\mathbb{Q}$ is elementarily equivalent to an ultraproduct of finite groups (cf. proof of Lemma 6.2(i)) it follows that Φ holds in some finite groups $\neq 1$. Case 2: $H\neq Z(H)$. Then $H/Z(G)$ is a locally normal group such that $H/Z(H) \models \Phi$. If $H/Z(H)$ is abelian, then Φ holds in some finite abelian group $\neq 1$ (this follows as in case 1). If $H/Z(H)$ is non-abelian, then there is a finite normal subgroup N in $H/Z(H)$ such that Φ holds in the non-trivial factor group $(H/Z(H))/C_{H/Z(H)}(N)$ which is finite, Q.E.D. ■

REMARK. Similarly as above one can prove that *Π_2-sentences which are true in all finite nilpotent (soluble) groups, are true in all locally-nilpotent (locally-soluble) torsion groups.* Notice that these groups are locally-finite.

In view of the results obtained so far we may say that the first-order theory Th(FC) of all FC-groups is not so far removed from the theory Th(Fin) of all finite groups, and that Th(FC) is rather distinct from the theory Th(G) of all groups. The theories Th(FC), Th(BFC) and Th(F-G/Z) seem to be very close to each other. In the sequel we are concerned with these theories.

Following M. Hall, Jr. and J.K. Senior a group G is called *capable* if there is a group H such that $G \cong H/Z(H)$.

Let G be a p-group for some prime p. An element $g \in G$ has *height* n if $g=h^{p^n}$ for

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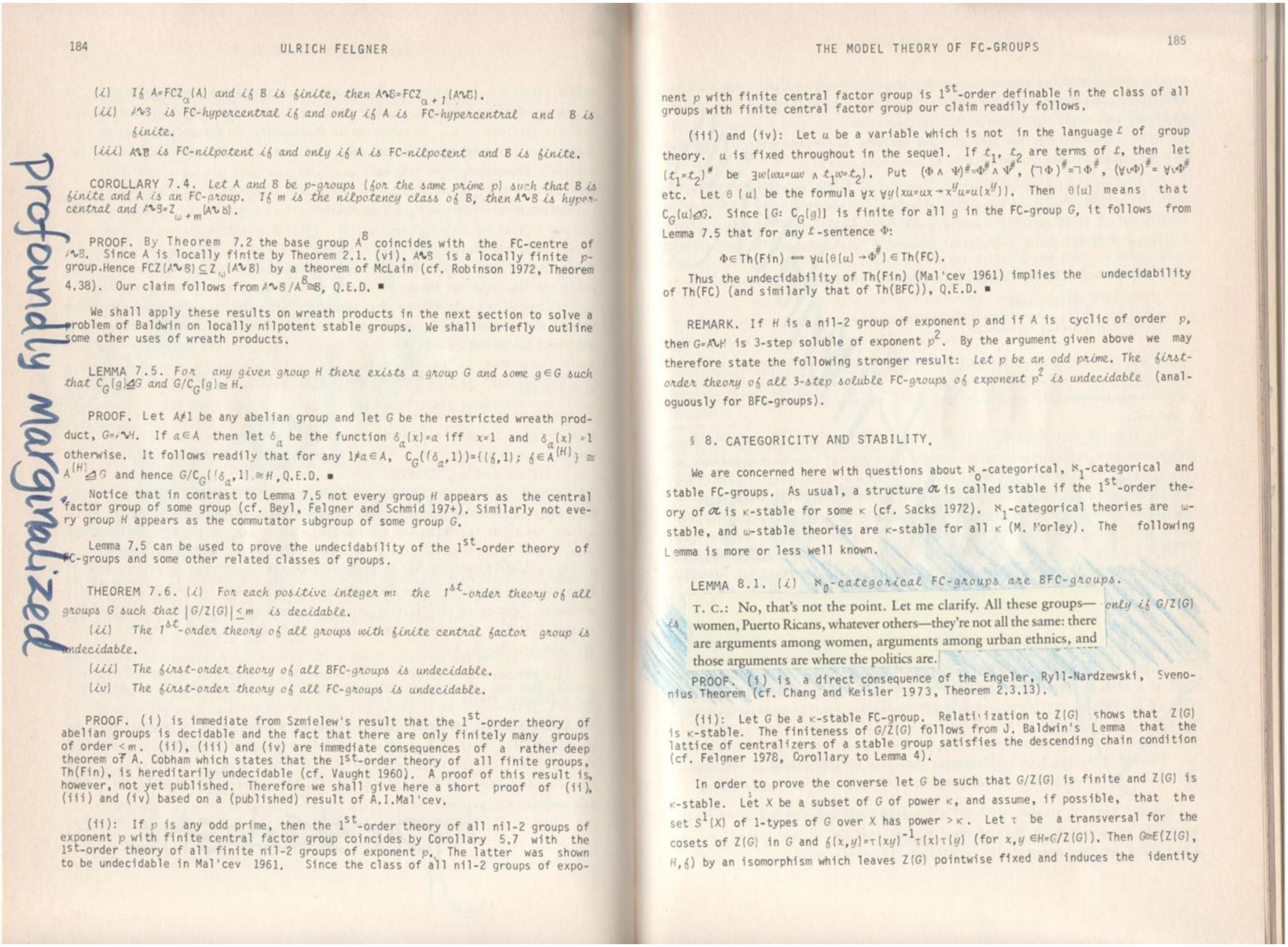

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(*i*) *If* $A = FCZ_\alpha(A)$ *and if B is finite, then* $A \wr B = FCZ_{\alpha+1}(A \wr B)$.

(*ii*) $A \wr B$ *is FC-hypercentral if and only if A is FC-hypercentral and B is finite.*

(*iii*) $A \wr B$ *is FC-nilpotent if and only if A is FC-nilpotent and B is finite.*

COROLLARY 7.4. *Let A and B be p-groups (for the same prime p) such that B is finite and A is an FC-group. If m is the nilpotency class of B, then* $A \wr B$ *is hypercentral and* $A \wr B = Z_{\omega+m}(A \wr B)$.

PROOF. By Theorem 7.2 the base group A^B coincides with the FC-centre of $A \wr B$. Since A is locally finite by Theorem 2.1. (vi), $A \wr B$ is a locally finite p-group. Hence $FCZ(A \wr B) \subseteq Z_\omega(A \wr B)$ by a theorem of McLain (cf. Robinson 1972, Theorem 4.38). Our claim follows from $A \wr B / A^B \cong B$, Q.E.D. ■

We shall apply these results on wreath products in the next section to solve a problem of Baldwin on locally nilpotent stable groups. We shall briefly outline some other uses of wreath products.

LEMMA 7.5. *For any given group H there exists a group G and some* $g \in G$ *such that* $C_G(g) \trianglelefteq G$ *and* $G/C_G(g) \cong H$.

PROOF. Let $A \neq 1$ be any abelian group and let G be the restricted wreath product, $G = A \wr H$. If $a \in A$ then let δ_a be the function $\delta_a(x) = a$ iff $x = 1$ and $\delta_a(x) = 1$ otherwise. It follows readily that for any $1 \neq a \in A$, $C_G((\delta_a, 1)) = \{(f, 1);\ f \in A^{(H)}\} \cong A^{(H)} \trianglelefteq G$ and hence $G/C_G((\delta_a, 1)) \cong H$, Q.E.D. ■

Notice that in contrast to Lemma 7.5 not every group H appears as the central factor group of some group (cf. Beyl, Felgner and Schmid 197+). Similarly not every group H appears as the commutator subgroup of some group G.

Lemma 7.5 can be used to prove the undecidability of the 1^{st}-order theory of FC-groups and some other related classes of groups.

THEOREM 7.6. (*i*) *For each positive integer m: the* 1^{st}*-order theory of all groups G such that* $|G/Z(G)| \leq m$ *is decidable.*

(*ii*) *The* 1^{st}*-order theory of all groups with finite central factor group is undecidable.*

(*iii*) *The first-order theory of all BFC-groups is undecidable.*

(*iv*) *The first-order theory of all FC-groups is undecidable.*

PROOF. (i) is immediate from Szmielew's result that the 1^{st}-order theory of abelian groups is decidable and the fact that there are only finitely many groups of order $\leq m$. (ii), (iii) and (iv) are immediate consequences of a rather deep theorem of A. Cobham which states that the 1^{st}-order theory of all finite groups, Th(Fin), is hereditarily undecidable (cf. Vaught 1960). A proof of this result is, however, not yet published. Therefore we shall give here a short proof of (ii), (iii) and (iv) based on a (published) result of A.I.Mal'cev.

(ii): If p is any odd prime, then the 1^{st}-order theory of all nil-2 groups of exponent p with finite central factor group coincides by Corollary 5.7 with the 1^{st}-order theory of all finite nil-2 groups of exponent p. The latter was shown to be undecidable in Mal'cev 1961. Since the class of all nil-2 groups of exponent p with finite central factor group is 1^{st}-order definable in the class of all groups with finite central factor group our claim readily follows.

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(iii) and (iv): Let u be a variable which is not in the language $\mathcal{L}$ of group theory. u is fixed throughout in the sequel. If t_1, t_2 are terms of $\mathcal{L}$, then let $(t_1 = t_2)^\# $ be $\exists w(wu = uw \wedge t_1 w = t_2)$. Put $(\Phi \wedge \Psi)^\# = \Phi^\# \wedge \Psi^\#$, $(\neg\Phi)^\# = \neg\Phi^\#$, $(\forall v \Phi)^\# = \forall v \Phi^\#$ etc. Let $\theta(u)$ be the formula $\forall x \forall y (xu = ux \rightarrow x^y u = u(x^y))$. Then $\theta(u)$ means that $C_G(u) \trianglelefteq G$. Since $[G : C_G(g)]$ is finite for all g in the FC-group G, it follows from Lemma 7.5 that for any $\mathcal{L}$-sentence Φ:

$$\Phi \in Th(Fin) \Longleftrightarrow \forall u(\theta(u) \rightarrow \Phi^\#) \in Th(FC).$$

Thus the undecidability of Th(Fin) (Mal'cev 1961) implies the undecidability of Th(FC) (and similarly that of Th(BFC)), Q.E.D. ■

REMARK. If H is a nil-2 group of exponent p and if A is cyclic of order p, then $G = A \wr H$ is 3-step soluble of exponent p^2. By the argument given above we may therefore state the following stronger result: *Let p be an odd prime. The first-order theory of all 3-step soluble FC-groups of exponent* p^2 *is undecidable* (analogously for BFC-groups).

§ 8. CATEGORICITY AND STABILITY.

We are concerned here with questions about $\aleph_0$-categorical, $\aleph_1$-categorical and stable FC-groups. As usual, a structure $\mathfrak{A}$ is called stable if the 1^{st}-order theory of $\mathfrak{A}$ is κ-stable for some κ (cf. Sacks 1972). $\aleph_1$-categorical theories are ω-stable, and ω-stable theories are κ-stable for all κ (M. Morley). The following Lemma is more or less well known.

LEMMA 8.1. (*i*) $\aleph_0$*-categorical FC-groups are BFC-groups.*

T. C.: No, that's not the point. Let me clarify. All these groups— women, Puerto Ricans, whatever others—they're not all the same: there are arguments among women, arguments among urban ethnics, and those arguments are where the politics are.

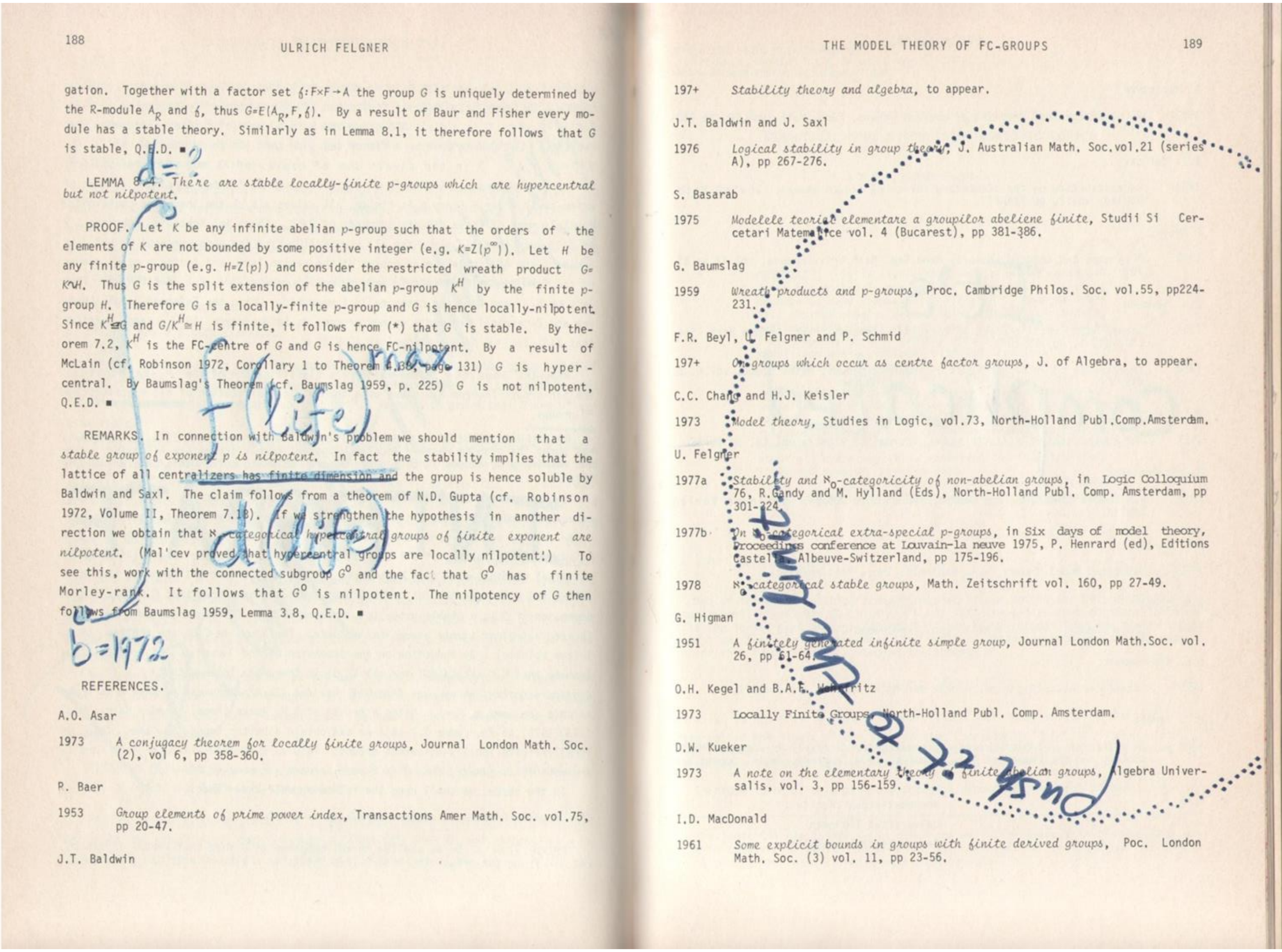
only if $G/Z(G)$ *is*

PROOF. (i) is a direct consequence of the Engeler, Ryll-Nardzewski, Svenonius Theorem (cf. Chang and Keisler 1973, Theorem 2.3.13).

(ii): Let G be a κ-stable FC-group. Relativization to $Z(G)$ shows that $Z(G)$ is κ-stable. The finiteness of $G/Z(G)$ follows from J. Baldwin's Lemma that the lattice of centralizers of a stable group satisfies the descending chain condition (cf. Felgner 1978, Corollary to Lemma 4).

In order to prove the converse let G be such that $G/Z(G)$ is finite and $Z(G)$ is κ-stable. Let X be a subset of G of power κ, and assume, if possible, that the set $S^1(X)$ of 1-types of G over X has power $> \kappa$. Let τ be a transversal for the cosets of $Z(G)$ in G and $f(x,y) = \tau(xy)^{-1}\tau(x)\tau(y)$ (for $x, y \in H = G/Z(G)$). Then $G \cong E(Z(G), H, f)$ by an isomorphism which leaves $Z(G)$ pointwise fixed and induces the identity

Mathematical Logic in Catholic North America, altered book, 2013, pp. 184-185.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gation. Together with a factor set $f: F\times F\to A$ the group G is uniquely determined by the R-module A_R and f, thus $G=E(A_R,F,f)$. By a result of Baur and Fisher every module has a stable theory. Similarly as in Lemma 8.1, it therefore follows that G is stable. Q.E.D. ■

d=?

LEMMA 8.4. *There are stable locally-finite p-groups which are hypercentral but not nilpotent.*

PROOF. Let K be any infinite abelian p-group such that the orders of the elements of K are not bounded by some positive integer (e.g. $K=Z(p^\infty)$). Let H be any finite p-group (e.g. $H=Z(p)$) and consider the restricted wreath product $G=K\wr H$. Thus G is the split extension of the abelian p-group K^H by the finite p-group H. Therefore G is a locally-finite p-group and G is hence locally-nilpotent. Since $K^H \trianglelefteq G$ and $G/K^H\cong H$ is finite, it follows from (*) that G is stable. By theorem 7.2, K^H is the FC-centre of G and G is hence FC-nilpotent. By a result of McLain (cf. Robinson 1972, Corollary 1 to Theorem 4.38, page 131) G is hyper-central. By Baumslag's Theorem (cf. Baumslag 1959, p. 225) G is not nilpotent, Q.E.D. ■

max

f(life)

REMARKS. In connection with Baldwin's problem we should mention that *a stable group of exponent p is nilpotent.* In fact the stability implies that the lattice of all centralizers has finite dimension and the group is hence soluble by Baldwin and Saxl. The claim follows from a theorem of N.D. Gupta (cf. Robinson 1972, Volume II, Theorem 7.18). If we strengthen the hypothesis in another direction we obtain that *$\aleph_0$-categorical hypercentral groups of finite exponent are nilpotent.* (Mal'cev proved that hypercentral groups are locally nilpotent!) To see this, work with the connected subgroup G^0 and the fact that G^0 has finite Morley-rank. It follows that G^0 is nilpotent. The nilpotency of G then follows from Baumslag 1959, Lemma 3.8, Q.E.D. ■

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b=1972

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